PHYS2170 Mathematical Methods 4

Problems Class 10

1. An elastic membrane is held around its edge by a rectangular frame that has a side of length a in the x-direction and of length b in the y-direction. The frame also holds the height $u(x, y)$ to be zero on all edges except for $y = 0$, upon which it is given by

$$u(x, 0) = \sin \frac{4\pi x}{a}.$$

The sheet satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Construct the solution using separation of variables:

(a) Write $u(x, y) = f(x)\, g(y)$. Determine the ordinary differential equations that f and g must obey.

(b) Choose $f(x)$ to satisfy the boundary conditions.

(c) Hence, solve the ODE for $g(y)$ and hence find the shape of the membrane, $u(x, y)$.

2. Consider the one dimensional Schrödinger equation,

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}.$$

Find all solutions in the form $\psi(x, t) = X(x)T(t)$, using separation of variables. Use E as the separation constant. Compare this to the solution of the diffusion equation (from lecture).

3. As a warm-up for Fourier Series, calculate the following integrals:

$$\int_0^{2\pi} \sin mx \, \sin nx \, dx, \qquad \int_0^{2\pi} \cos mx \, \sin nx \, dx,$$

where n, m are positive integers. Be careful about the cases where m and n are equal or not equal.