PHYS2170 Mathematical Methods 4

Problems Class 10

1. *An elastic membrane is held around its edge by a rectangular frame that has a side of length a in the x-direction and of length b in the y-direction. The frame also holds the height $u(x,y)$ to be zero on all edges except for $y=0$, upon which it is given by*

$$u(x,0) = \sin\frac{4\pi x}{a}.$$

(a) *Write $u(x,y) = f(x)\,g(y)$. Determine the ordinary differential equations that f and g must obey.*

Using separation of variables, direct substitution leads to

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\frac{1}{g}\frac{d^2g}{dy^2} = -k^2 \Longrightarrow \begin{cases} \frac{d^2f}{dx^2} = -k^2 f \\ \frac{d^2g}{dy^2} = k^2\,g \end{cases}$$

[Choose the sign because we anticipate the oscillatory character of $f(x)$, given the boundary conditions].

(b) *Choose $f(x)$ to satisfy the boundary conditions.*

The general solution for f is $f(x) = A\sin kx + B\cos kx$. To satisfy the boundary condition at $y=0$ requires $B=0, k=4\pi/a$. This will also automatically satisfy the boundary condition of $f(x)=0$ at $x=0$ and $x=a$.

(c) *Hence, solve the ODE for $g(y)$ and hence find the shape of the membrane, $u(x,y)$.*

Now we know that $g(y) = Ce^{ky} + De^{-ky}$. We require:

$$u(x,0) = \underbrace{\sin\frac{4\pi x}{a}}_{given b.c.} = f(x)g(0) = \sin\frac{4\pi x}{a}(C+D) \quad \Rightarrow \quad C+D=1$$

$$u(x,b) = 0 = f(x)g(b) = \sin\frac{4\pi x}{a}(Ce^{kb}+De^{-kb}) \quad \Rightarrow \quad Ce^{kb}+De^{-kb}=0$$

Solving yields

$$C = \frac{-e^{-kb}}{2\sinh kb}, D = \frac{e^{kb}}{2\sinh kb}$$

$$\Rightarrow u(x,y) = \frac{\sin\frac{4\pi x}{a}}{2\sinh kb}\left[e^{-k(y-b)} - e^{k(y-b)}\right] = \frac{\sin\frac{4\pi x}{a}\sinh k(y-b)}{\sinh kb}$$

2. Assuming $\psi(x,t) = X(x)T(t)$ yields

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} = \frac{i\hbar}{T}\frac{dT}{dt} = \pm|E|,$$

where E is the separation constant (the energy associated with the wavefunction), which we can consider to be either positive *or* negative.* Hence T satisfies a first order

*It *could* be assumed that E is a complex or imaginary number, and there are valid mathematical solutions in such a case. Physically, however, E is the energy and is taken to be a real number. Notice that if E is taken to be imaginary, the solutions will be exponential functions of time and exponential functions space, rather than both oscillatory.

ODE while X satisfies a second order ODE. The solutions are:

$$T(t) = Ae^{-i\frac{E}{\hbar}t}, \qquad X(x) = \begin{cases} B\sin kx + C\cos kx, & (E>0), \\ B\sinh kx + C\cosh kx, & (E<0), \end{cases}$$

where $k^2 = \frac{2m|E|}{\hbar^2}$. Hence, the wave function is

$$\psi(x,t) = \begin{cases} e^{-i\frac{E}{\hbar}t}\left[B\sin kx + C\cos kx\right], & (E>0) \\ e^{-i\frac{E}{\hbar}t}\left[B\sinh kx + C\cosh kx\right], & (E<0). \end{cases}$$

We could also have chosen eigenfunctions $e^{\pm kx}$ instead of $\cosh kx$ and $\sinh kx$. [The factors of $\hbar$ are chosen to be distributed this way so that E has units of energy.] Notice that, because of the i, the character of the solutions is different from the solutions to the superficially similar diffusion equation. In the diffusion equation the solutions are exponential in time, while the solutions to the Schrödinger equation are oscillatory in both time and space (for positive energy). Hence the Schrödinger equation is truly a wave equation! For negative energy $E = -|E|$, the spatial part of the wave function is exponential; this describes, for example, a particle tunnelling through a potential barrier.

3.

$$\begin{aligned}
\int_0^{2\pi} \sin mx\,\sin nx\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\cos(m-n)x - \cos(m+n)x\right] \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{2\pi} & (m\neq n) \\ \frac{1}{2}\left[x - \frac{\sin 2mx}{2m}\right]_0^{2\pi} & (m=n) \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin 2(m-n)\pi}{m-n} - \frac{\sin 2(m+n)\pi}{m+n}\right] = 0 & (m\neq n) \\ \frac{1}{2}\left[2\pi - \frac{\sin 4m\pi}{2m}\right] = \pi & (m=n) \end{cases} \\
\int_0^{2\pi} \cos mx\,\sin nx\,dx &= \frac{1}{2}\int_0^{2\pi}\left[\sin(m+n)x - \sin(m-n)x\right] \\
&= \begin{cases} \frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_0^{2\pi} = 0 & (m\neq n) \\ \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_0^{2\pi} = 0 & (m=n) \end{cases} \\
&= 0
\end{aligned}$$