PHYS2170 Mathematical Methods 4

Problems Class 11: Solutions

The Fourier Series for a function $f(x)$ with period L is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2n\pi x}{L}\right)dx, \quad b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2n\pi x}{L}\right)dx.$$

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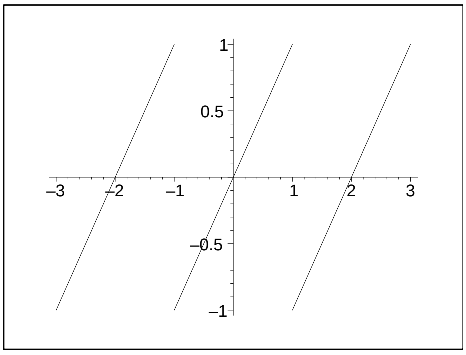
1. (a) $S(x) = x, -a < x < a$: The function is odd.



(b) Since it is odd there are only sin functions in the Fourier series. We must calculate (using $L = 2a$)

$$\begin{aligned} b_n &= \frac{1}{a}\int_{-a}^{a} x \sin\left(\frac{n\pi x}{a}\right)dx \\ &= \frac{1}{a}\left[-x\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\Big|_{-a}^{a} + \underbrace{\int_{-a}^{a}\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)dx}_{\to 0}\right] \\ &= -\left(\frac{2a}{n\pi}\right)\cos n\pi = (-1)^{n+1}\frac{2a}{n\pi}. \end{aligned}$$

Hence,

$$S(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2a}{n\pi}\sin\left(\frac{n\pi x}{a}\right).$$

(c) If we choose $x = a/2$, we have

$$\frac{a}{2} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2a}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2a}{n\pi} \times \begin{cases} 0, & n = 2, 4, 6, \ldots \\ +1, & n = 1, 5, 9, \ldots \\ -1, & n = 3, 7, 11, \ldots \end{cases}$$

Hence, the sum is over all odd n, which can be parametrized by $n = 2m+1$, with $m = 0, 1, 2, 3, \ldots$:

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} (-1)^{2m+1+1} \frac{1}{(2m+1)} \times \begin{cases} +1, & 2m+1 = 1, 5, 9, \ldots \Rightarrow m = 0, 2, 4, \ldots \\ -1, & 2m+1 = 3, 7, 11, \ldots \Rightarrow m = 1, 3, 5, \ldots \end{cases}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)}$$

2. *Now consider an elastic sheet whose height $u(x, y)$ obey's Laplace's equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

on the domain $0 < x < a$, $0 < y < b$, with boundary conditions

$$u(0, y) = u(a, y) = 0, \qquad u(x, b) = 0, \qquad u(x, 0) = S(x).$$

The boundary condition on x is $S(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)$.

(a) Separating variables for each Fourier mode, we recall that a general solution is given by $u_n(x, y) = b_n(y)\sin(k_n x)$, where we anticipate that $b_n(y)$ will be exponential, since the x-dependence is oscillatory. The general solution will be written as the superposition

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin\left(\frac{n\pi x}{a}\right).$$

with $k_n = n\pi/a$. Substituting the assumption converts Laplace's equation (for *each* Fourier mode) to:

$$0 = -k_n^2 \sin k_n x \, b_n(y) + \sin k_n x \, b_n''(y)$$
$$k_n^2 \, b_n(y) = b_n''(y)$$
$$\Longrightarrow b_n(y) = A_n e^{k_n y} + B_n e^{-k_n y}$$

$$u_n(x, y) = \begin{cases} \left(A_n e^{k_n y} + B_n e^{-k_n y}\right) \sin k_n x. \\ \left(C_n \sinh k_n(y-b) + D_n \cosh k_n(y-b)\right) \sin k_n x. \end{cases}$$

Either of the last two representations is correct; which is used depends on convenience [recall that $\sinh x = (e^x - e^{-x})/2, \cosh x = (e^x + e^{-x})/2$].

(b) The boundary conditions $u(0, y) = u(a, y)$ are automatically satisfied by the $\sin k_n x$ dependence of each mode. To satisfy the boundary conditions at $y = 0, b$ requires, *for each mode*,

$$u(x, b) = 0, \qquad \Longrightarrow \qquad A_n e^{k_n b} + B_n e^{-k_n b} = 0$$
$$u(x, 0) = S(x) \qquad \Longrightarrow \qquad A_n + B_n = (-1)^{n+1} \frac{2a}{n\pi}.$$

or

$$u(x,b) = 0, \qquad \Longrightarrow \qquad D_n = 0$$
$$u(x,0) = S(x) \qquad \Longrightarrow \qquad -C_n \sinh k_n b + D_n \cosh k_n b = (-1)^{n+1} \frac{2a}{n\pi}.$$

Each mode has to separately satisfy a relation because the basis functions $\sin k_n x$ are all *independent* of each other. We can see that in this case it's slightly easier to deal with the hyperbolic functions; $D_n = 0$, and C_n is given by

$$C_n = \frac{(-1)^n 2a}{n\pi \sinh k_n b}$$

$$\Longrightarrow \qquad \boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2a}{n\pi} \frac{\sinh k_n (b-y)}{\sinh k_n b} \sin k_n x, \qquad (k_n = \frac{n\pi}{a})}$$