PHYS2170 Mathematical Methods 4

Problems Class 1: Solutions

1. Differentiate the following functions:

   (a) $\frac{\partial}{\partial x}(xe^{ax}) = e^{ax} + axe^{ax}$

   (b) $\frac{\partial}{\partial x}(\sin(x^2)) = (2x)\cos(x^2)$

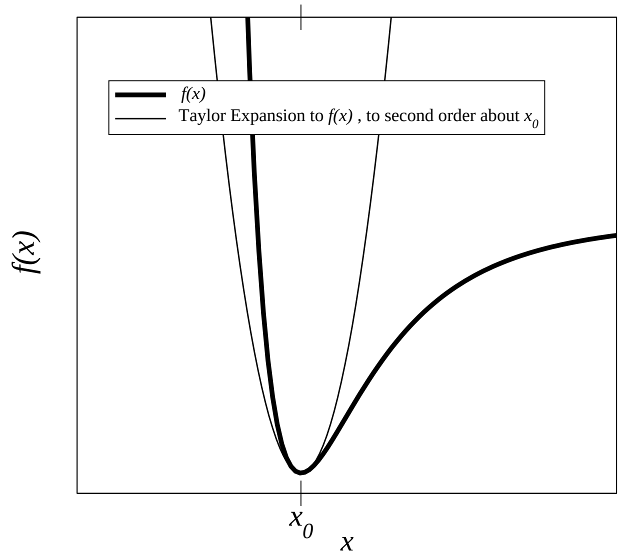
   (c)
$$\frac{\partial}{\partial x}\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right) = \frac{1}{2}\frac{1}{\sqrt{1+x}\sqrt{1-x}} + \left(-\frac{1}{2}\right)\frac{-\sqrt{1+x}}{(1-x)^{3/2}}$$
$$= \frac{1}{2}\left[\frac{1-x}{\sqrt{1+x}(1-x)^{3/2}} + \frac{(\sqrt{1+x})^2}{\sqrt{1+x}(1-x)^{3/2}}\right] = \frac{1}{\sqrt{1+x}(1-x)^{3/2}}$$

2. Calculate the Taylor Series up to second order terms for

   (a) $e^{ax} \simeq e^{a\cdot 0} + ae^{a\cdot 0}\,x + \frac{a^2}{2!}e^{a\cdot 0}\,x^2 \ldots = 1 + a\,x + \frac{1}{2!}a^2\,x^2 + \ldots.$

   (b) $1/x \simeq \dfrac{1}{r} - \dfrac{1}{r^2}(x-r) + \dfrac{1}{r^3}(x-r)^2 + \ldots$ (Coulomb potential).

3. (a) A sketch is as below:



   (b) For $f(x) = \dfrac{A}{x^{12}} - \dfrac{B}{x^6}$, the minimum is given by $\frac{\partial f}{\partial x}\big|_{x_0} = 0 = -12\dfrac{A}{x_0^{13}} + 6\dfrac{B}{x_0^7} = 0 \Longrightarrow x_0^6 = 2A/B$, or $x_0 = (2A/B)^{1/6}$.

   (c) The Taylor expansion is given by
$$f \simeq f(x_0) + \frac{\partial f}{\partial x}\bigg|_{x_0}(x-x_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bigg|_{x_0}(x-x_0)^2 + \ldots$$
$$= A\left(\tfrac{B}{2A}\right)^2 - B\tfrac{B}{2A} + 0 + \tfrac{1}{2}\left[\tfrac{12\cdot 13A}{x^{14}} - \tfrac{6\cdot 7B}{x^8}\right]_{x_0}(x-x_0)^2.$$
$$= -\tfrac{B^2}{4A} + 9\cdot 2^{2/3}A^{-4/3}B^{7/3}(x-x_0)^2.$$

   It's a parabola (harmonic oscillator potential!!) centred at x_0. Note that you don't have to recalculate the linear term, $\frac{\partial f}{\partial x}\big|_{x_0}$, because x_0 is *defined* so that this vanishes!

(d) The function $f(x)$ is the van der Waals intermolecular potential energy due to dipolar fluctuations. The Taylor expansion is useful for understanding fluctuations of, for example, solids (lattice vibrations) around the local minima.

4. For $\phi(\mathbf{r}) = \frac{1}{\sqrt{x^2+y^2+z^2}}$:

(a)

$$\begin{aligned}\boldsymbol{\nabla}\phi &= -\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{3/2}}\left(2x\widehat{\boldsymbol{i}} + 2y\widehat{\boldsymbol{j}} + 2z\widehat{\boldsymbol{z}}\right) \\ &= -\frac{1}{(x^2+y^2+z^2)^{3/2}}\left(x\widehat{\boldsymbol{i}} + y\widehat{\boldsymbol{j}} + z\widehat{\boldsymbol{z}}\right) = -\frac{1}{r^3}\mathbf{r} = -\frac{1}{r^2}\widehat{\boldsymbol{r}},\end{aligned}$$

where $\widehat{\boldsymbol{r}}$ is the unit vector in the direction of $\mathbf{r}$ (any of the forms above are correct!); and recall that $r^2 = x^2 + y^2 + z^2$.

(b) Using the standard determinant definition of the vector product:

$$\mathbf{r} \times \boldsymbol{\nabla}\phi = -\frac{1}{r^3}\begin{vmatrix} \widehat{\boldsymbol{i}} & \widehat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ x & y & z \\ x & y & z \end{vmatrix} = 0.$$

Or, this is easier because $\mathbf{r} \times \boldsymbol{\nabla}\phi = (-1/r^3)\mathbf{r} \times \mathbf{r} = 0$, using the properties that a vector products of a vector with itself vanishes.

5.

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \widehat{\boldsymbol{i}} & \widehat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x\sin z & -z^2 \end{vmatrix} \\ &= \widehat{\boldsymbol{i}}(0 - x\cos z) + \widehat{\boldsymbol{j}}(0-0) + \widehat{\boldsymbol{k}}(\sin z - x) = -x\cos z\ \widehat{\boldsymbol{i}} + (\sin z - x)\widehat{\boldsymbol{k}}.\end{aligned}$$