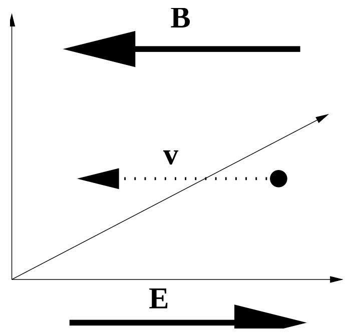
PHYS2170 Mathematical Methods 4

Problems Class 2: Solutions

1. The equation of motion is $m\dot{\mathbf{v}} = q\left[\mathbf{v} \times \mathbf{B} + \mathbf{E}\right]$, where $q = -e$ and $\mathbf{E} = E_0\hat{\mathbf{x}}$.

   (a) At time $t = 0$ the particle is at rest. Hence it is first accelerated in the direction anti-parallel to the electric field $\mathbf{E}$ (anti-parallel because of the negative charge), because the $\mathbf{v} \times \mathbf{B}$ term is zero. Once it starts to move the first term can have an effect. If $\mathbf{v} \times \mathbf{B} = 0$ the magnetic field will have no effect and the particle will continue to move antiparallel to $\mathbf{E}$. This will be the case for <u>**B** parallel (or anti-parallel) to **E**.</u>



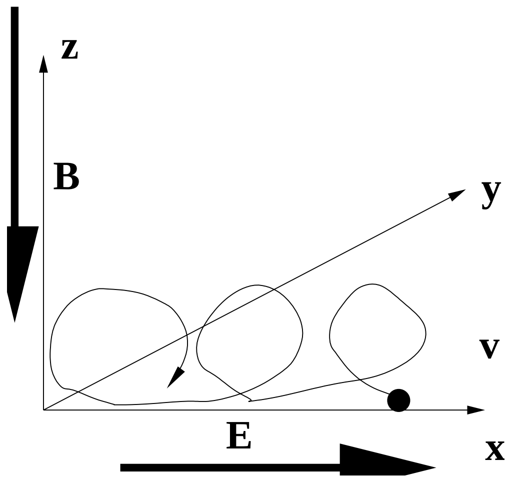
   (b) The electric field will accelerate the particle in the $-\widehat{\boldsymbol{i}}$, anti-parallel to the x direction. If the magnetic field is parallel or anti-parallel to the z-axis, $\mathbf{B} = \pm B_0\widehat{\boldsymbol{k}}$, then by the right hand rule the motion will only be in the $x - y$ plane. We must choose the sign of $\mathbf{B}$ so that, once the particle moves, it moves in the positive y direction, $+\widehat{\boldsymbol{j}}$. After a small time dt the velocity will be

$$\mathbf{v} = -\tfrac{1}{m}eE_0\, dt\widehat{\boldsymbol{i}}.$$

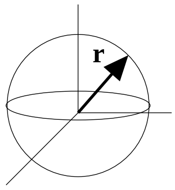
   For $\mathbf{B} = B_0\widehat{\boldsymbol{k}}$ the acceleration due to the magnetic field will be

$$q\mathbf{v} \times \mathbf{B} = \tfrac{1}{m}(-e)(-eE_0dt)\widehat{\boldsymbol{i}} \times (B_0\widehat{\boldsymbol{k}}) = \tfrac{1}{m}e^2E_0B_0\, dt\widehat{\boldsymbol{i}} \times \widehat{\boldsymbol{k}} = -\tfrac{1}{m}e^2E_0B_0\widehat{\boldsymbol{j}}.$$

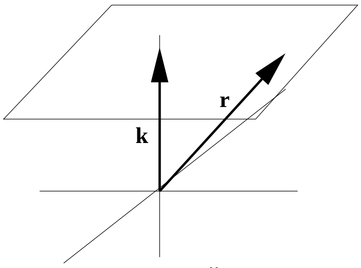
   This is the negative y direction. Hence we need $\mathbf{B} = -B_0\widehat{\boldsymbol{k}}$, in the negative z direction, to have motion in the $x - y$ plane with $y > 0$.



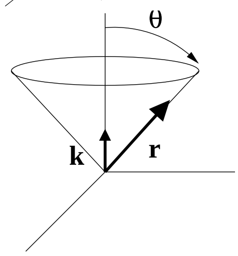
2.



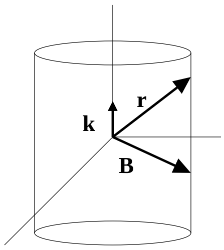
(a) $|\mathbf{r}| = k$. This is a sphere with radius k.



(b) $\mathbf{r}\cdot\widehat{\boldsymbol{k}} = \ell$, where $\widehat{\boldsymbol{k}}$ is the unit vector along the z-axis. This is an infinite plane parallel to the $x-y$ plane, a distance ℓ from the origin in the z direction.



(c) $\mathbf{r}\cdot\widehat{\boldsymbol{k}} = m|\mathbf{r}|$. From the definition of the scalar product, $\mathbf{r}\cdot\widehat{\boldsymbol{k}} = |r|\cos\theta$, where θ is the angle with respect to the z axis. Hence the surface is all points that lie at the angle $\theta = \arccos(m)$; *i. e.* a cone.



(d) $|\mathbf{r} - (\mathbf{r}\cdot\widehat{\boldsymbol{k}})\widehat{\boldsymbol{k}}| = n$. This is the set of points such that the magnitude of the vector $\mathbf{r}-(\mathbf{r}\cdot\widehat{\boldsymbol{k}})\widehat{\boldsymbol{k}}$ is a constant. What is this vector? It is obtained by subtracting from $\mathbf{r}$ the portion of the vector that is parallel to the z axis. Recall that we can resolve

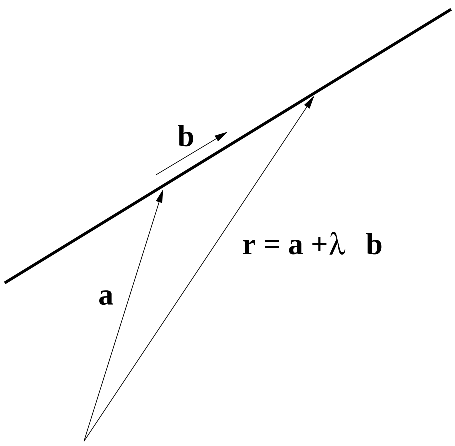
$$\mathbf{r} = A\widehat{\boldsymbol{k}} + \mathbf{B},$$

where $A = \mathbf{r}\cdot\widehat{\boldsymbol{k}}$ and $\mathbf{B}$ is orthogonal to $\widehat{\boldsymbol{k}}$, and is in fact the vector we are looking for. Hence (see figure) the desired surface is the surface of a cylinder of radius n.

3. To show that the set of vectors $(1,0,1), (1,1,0)$, and $(1,-3,4)$ lie on a line, we must examine the differences between them. For example,

$$\begin{aligned}(1,0,1)-(1,1,0) &= (0,-1,1)\\ (1,0,1)-(1,-3,4) &= (0,3,-3)\\ (1,1,0)-(1,-3,4) &= (0,4,-4).\end{aligned}$$

The difference between any two of them is a scalar multiple of the same vector, $(0,1,-1)$. Hence they can be reached by moving parallel to a single vector, and they lie on a straight line.
The equation of the line can be given as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is the independent variable parametrizing points on the line. $\mathbf{a}$ is any point on the line, and $\mathbf{b}$ is the direction one must travel to reach other points on the line. We can choose $\mathbf{b} = (0,1,-1)$ (or any scalar multiple), and $\mathbf{a}$ any of the points we are given. Hence the following are all equivalent representations of the line:



$$
\begin{aligned}
\mathbf{r} &= (1,0,1) + \lambda(0,1,-1) \\
\mathbf{r} &= (1,1,0) + \lambda(0,1,-1) \\
\mathbf{r} &= (1,-3,4) + \lambda(0,1,-1).
\end{aligned}
$$