PHYS2170 Mathematical Methods 4

Problems Class 3

1. Compute the following, leaving the answer in a coordinate-free form (no x, y, or z, for example). In each case, $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

   (a) $\boldsymbol{\nabla} \sin(\mathbf{q} \cdot \mathbf{r})$.

   (b) $\boldsymbol{\nabla} \times \mathbf{r}$.

   (c) $\boldsymbol{\nabla} e^{-r/\lambda}$, where $r = |\mathbf{r}|$.

2. Let
$$\phi = \mathbf{B} \cdot \mathbf{r} + (\mathbf{C} \cdot \mathbf{r})^2 ,$$
   with $\mathbf{B} = (1, -1) = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{C} = (2, 2) = 2(\hat{\mathbf{i}} + \hat{\mathbf{j}})$.

   (a) Compute: (i) $\boldsymbol{\nabla}\phi$ (ii) $\mathbf{B} \times \boldsymbol{\nabla}\phi$, and (iii) $\mathbf{C} \cdot \mathbf{B} \times \boldsymbol{\nabla}\phi$ entirely in general vector form in terms of $\mathbf{B}, \mathbf{C}$, and $\mathbf{r}$.
   [Hint: after you've computed $\boldsymbol{\nabla}\phi$, write $\boldsymbol{\nabla}\phi$ only in terms of $\mathbf{B}, \mathbf{C}$, and use that for the rest of the problem in place of $\boldsymbol{\nabla}\phi$]

   (b) Compute $\boldsymbol{\nabla}\phi$ in Cartesian coordinates.

3. Demonstrate the following equality:
$$\boldsymbol{\nabla} \times [\mathbf{A}\,(\boldsymbol{\nabla} \cdot \mathbf{A})] + \mathbf{A} \times [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})] + \mathbf{A} \times \nabla^2 \mathbf{A} = (\boldsymbol{\nabla} \cdot \mathbf{A})\, \boldsymbol{\nabla} \times \mathbf{A}$$
   where ϕ and $\mathbf{A}$ both depend on $\mathbf{r}$.
   You will need the identities:
   (1) $\boldsymbol{\nabla} \times (\phi \mathbf{V}) = \phi \boldsymbol{\nabla} \times \mathbf{V} + (\boldsymbol{\nabla}\phi) \times \mathbf{V}$
   (2) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$.