PHYS2170 Mathematical Methods 4

Problems Class 3: Solutions

1. (a) $\boldsymbol{\nabla}\sin(\mathbf{q}\cdot\mathbf{r}) = \cos(\mathbf{q}\cdot\mathbf{r})\boldsymbol{\nabla}(\mathbf{q}\cdot\mathbf{r}) = \mathbf{q}\cos(\mathbf{q}\cdot\mathbf{r})$ (using the fact that $\boldsymbol{\nabla}(\mathbf{q}\cdot\mathbf{r}) = \mathbf{q}$).

   (b) $\boldsymbol{\nabla}\times\mathbf{r} = 0$ (You need coordinates for this; just calculate the curl in the usual way using $\mathbf{r} = x\,\widehat{\boldsymbol{i}} + y\,\widehat{\boldsymbol{j}} + z\,\widehat{\boldsymbol{k}}$). Alternatively, if you plot the *vector field* $\mathbf{r}$, you will see that it is radial and can't possibly have a curl!

   (c) $\boldsymbol{\nabla}e^{-r/\lambda} = \boldsymbol{\nabla}e^{-\sqrt{\mathbf{r}\cdot\mathbf{r}}/\lambda} = -\frac{1}{\lambda}e^{-r/\lambda}\frac{1}{2}(\mathbf{r}\cdot\mathbf{r})^{-1/2}\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{r}) = -e^{-r/\lambda}\frac{1}{2\lambda}\frac{1}{r}2\mathbf{r} = -\frac{\widehat{\boldsymbol{r}}}{\lambda}e^{-r/\lambda}$.
   Alternatively (or equivalently):
   $\boldsymbol{\nabla}e^{-r/\lambda} = e^{-r/\lambda}(-\frac{1}{\lambda})\boldsymbol{\nabla}(r) = -\frac{1}{\lambda}e^{-r/\lambda}\boldsymbol{\nabla}(\sqrt{\mathbf{r}\cdot\mathbf{r}}) = -\frac{1}{\lambda}e^{-r/\lambda}\frac{1}{2}(\mathbf{r}\cdot\mathbf{r})^{-1/2}\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{r}) = -e^{-r/\lambda}\frac{1}{2\lambda}\frac{1}{r}2\mathbf{r} = -\frac{\widehat{\boldsymbol{r}}}{\lambda}e^{-r/\lambda}$.

2. $\phi = \mathbf{B}\cdot\mathbf{r} + (\mathbf{C}\cdot\mathbf{r})^2$, with $\mathbf{B} = (1,-1)$ and $\mathbf{C} = (2,2)$.

   (a)
   - $\boldsymbol{\nabla}\phi = \mathbf{B} + 2(\mathbf{C}\cdot\mathbf{r})\boldsymbol{\nabla}(\mathbf{C}\cdot\mathbf{r}) = \mathbf{B} + 2(\mathbf{C}\cdot\mathbf{r})\,\mathbf{C}$ (use chain rule!).
   - $\mathbf{B}\times\boldsymbol{\nabla}\phi = \mathbf{B}\times[\mathbf{B} + 2(\mathbf{C}\cdot\mathbf{r})\,\mathbf{C}] = 0 + 2(\mathbf{C}\cdot\mathbf{r})(\mathbf{B}\times\mathbf{C}) = 2(\mathbf{C}\cdot\mathbf{r})(\mathbf{B}\times\mathbf{C})$.
   - $\mathbf{C}\cdot\mathbf{B}\times\boldsymbol{\nabla}\phi = 2\mathbf{C}\cdot(\mathbf{C}\cdot\mathbf{r})(\mathbf{B}\times\mathbf{C}) = 2(\mathbf{C}\cdot\mathbf{r})\,\mathbf{C}\cdot(\mathbf{B}\times\mathbf{C}) = 0$ (since $\mathbf{B}\times\mathbf{C}$ is orthogonal to $\mathbf{C}$ and their scalar product therefore vanishes).

   (b) $\boldsymbol{\nabla}\phi = \widehat{\boldsymbol{i}} - \widehat{\boldsymbol{j}} + 2(2x+2y)(2\widehat{\boldsymbol{i}} + 2\widehat{\boldsymbol{j}})$ (bit of a mess).

3. Simplify the following expression,

$$
\begin{aligned}
&\boldsymbol{\nabla}\times[\mathbf{A}\,(\boldsymbol{\nabla}\cdot\mathbf{A})] + \mathbf{A}\times[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})] + \mathbf{A}\times\nabla^2\mathbf{A} \\
&= \underbrace{(\boldsymbol{\nabla}\cdot\mathbf{A})\,\boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\mathbf{A})\times\mathbf{A}}_{\text{Identity (1)}} + \underbrace{\mathbf{A}\times\left[\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\right]}_{\text{Identity (2)}} + \mathbf{A}\times\nabla^2\mathbf{A} \\
&= (\boldsymbol{\nabla}\cdot\mathbf{A})\,\boldsymbol{\nabla}\times\mathbf{A} \quad \underbrace{-\mathbf{A}\times\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\mathbf{A})}_{\text{Reverse vector product, change sign}} \quad +\mathbf{A}\times\,\boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\mathbf{A}) - \mathbf{A}\times\nabla^2\mathbf{A} + \mathbf{A}\times\nabla^2\mathbf{A} \\
&= (\boldsymbol{\nabla}\cdot\mathbf{A})\,\boldsymbol{\nabla}\times\mathbf{A}.
\end{aligned}
$$

$$
\text{Identities:}\quad \begin{cases} (1) \quad \boldsymbol{\nabla}\times(\phi\mathbf{V}) = \phi\boldsymbol{\nabla}\times\mathbf{V} + (\boldsymbol{\nabla}\phi)\times\mathbf{V} \\ (2) \quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V}) - \nabla^2\mathbf{V} \end{cases}
$$