PHYS2170 Mathematical Methods 4

Problems Class 4

1. The electric field in a propagating electromagnetic wave (*e.g.* light!) is given by

$$\mathbf{E} = \mathbf{E}_0 \cos\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right), \tag{1}$$

where $\mathbf{k}$ is a constant "wavevector" and $\mathbf{E}_0$ is a constant (vector) amplitude.

   (a) Calculate $\boldsymbol{\nabla}\cdot\mathbf{E}$. Given Maxwell's equation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ in free space, what does this tell you about the relation between the direction of the polarization $\mathbf{E}_0$ and the direction of travel $\mathbf{k}$ of the wave?

   (b) Calculate $\boldsymbol{\nabla}\times\mathbf{E}$.

   (c) Calculate $\mathbf{k}\cdot\boldsymbol{\nabla}\times\mathbf{E}$.

   [Recall that $\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) = \mathbf{A}\cdot\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times(\phi\mathbf{A}) = (\boldsymbol{\nabla}\phi)\times\mathbf{A} + \phi\boldsymbol{\nabla}\times\mathbf{A}$].

2. Consider a force $\mathbf{F}$ and a momentum $\mathbf{p}$ given (in 2 dimensions) by

$$\mathbf{F} = \frac{x\, e^{-(x^2+y^2)}}{x^2+y^2}\widehat{\boldsymbol{i}} + \frac{y\, e^{-(x^2+y^2)}}{x^2+y^2}\widehat{\boldsymbol{j}} \qquad \mathbf{p} = \frac{y\,\widehat{\boldsymbol{i}}}{(x^2+y^2)^{3/2}} - \frac{x\,\widehat{\boldsymbol{j}}}{(x^2+y^2)^{3/2}}. \tag{2}$$

   (a) Write $\mathbf{F}$ and $\mathbf{p}$ entirely in polar coordinates ($x = \rho\cos\phi, y = \rho\sin\phi$, $\boldsymbol{\rho} = x\,\widehat{\boldsymbol{i}} + y\,\widehat{\boldsymbol{j}}$, $\widehat{\boldsymbol{\phi}} = -\sin\phi\widehat{\boldsymbol{i}} + \cos\phi\widehat{\boldsymbol{j}}, \ldots$)

   (b) Compute $\boldsymbol{\nabla}\cdot\mathbf{F}$ and $\boldsymbol{\nabla}\cdot\mathbf{p}$ in polar coordinates ($\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi}$).

   (c) Calculate $\mathbf{F}\times\mathbf{p}$. You can perform this using Cartesian coordinates, or directly and more easily by noticing the directions of $\mathbf{F}$ and $\mathbf{p}$ in polar coordinates (you will have to introduce the third dimension!).

3. Evaluate

$$I = 2\int_S dA\,(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{A}),$$

where $\mathbf{A} = (1, 2)$, and S is a quarter of the circle of radius 2 centered at the origin, with $\{x > 0, y > 0\}$ [Recall the area element $dA = r dr d\theta$ in polar coordinates, $x = r\cos\theta, y = r\sin\theta$]. This is easiest to evaluate using polar coordinates.