**PHYS2170 Mathematical Methods 4**

Problems Class 4: Solutions

1. The electric field in a propagating electromagnetic wave (*e.g.* light!) is given by

$$\mathbf{E} = \mathbf{E}_0 \cos\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right), \tag{1}$$

where $\mathbf{k}$ is a constant "wavevector". This revisits the example we considered in lecture.

(a)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= \boldsymbol{\nabla}\cdot(\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)) = \mathbf{E}_0\cdot\boldsymbol{\nabla}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\\ &= \mathbf{E}_0\cdot(-\sin(\mathbf{k}\cdot\mathbf{r}-\omega t))\boldsymbol{\nabla}(\mathbf{k}\cdot\mathbf{r}) = \mathbf{E}_0\cdot(-\sin(\mathbf{k}\cdot\mathbf{r}-\omega t))\mathbf{k}\\ &= -\mathbf{k}\cdot\mathbf{E}_0\sin(\mathbf{k}\cdot\mathbf{r}-\omega t).\end{aligned}$$

Maxwell's equation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ in free space thus implies $\mathbf{k}\cdot\mathbf{E}_0 = 0$ (since it must be true for all time and space). Hence, the polarization of the elecric field is perpendicular to the direction of propagation.

(b) $\boldsymbol{\nabla}\times\mathbf{E} = [\boldsymbol{\nabla}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)]\times\mathbf{E}_0 = [-\mathbf{k}\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)]\times\mathbf{E}_0 = -\mathbf{k}\times\mathbf{E}_0\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)$.

(c) $\mathbf{k}\cdot\boldsymbol{\nabla}\times\mathbf{E} = [\mathbf{k}\cdot(-\mathbf{k}\times\mathbf{E}_0)]\sin(\mathbf{k}\cdot\mathbf{r}-\omega t) = 0$.

2. (a) In polar coordinates,

$$\mathbf{F} = \frac{x\,e^{-(x^2+y^2)}}{x^2+y^2}\widehat{\boldsymbol{i}} + \frac{y\,e^{-(x^2+y^2)}}{x^2+y^2}\widehat{\boldsymbol{j}} = \boldsymbol{\rho}\frac{e^{-\rho^2}}{\rho^2} = \widehat{\boldsymbol{\rho}}\frac{e^{-\rho^2}}{\rho} \tag{2}$$

$$\mathbf{p} = \frac{y\,\widehat{\boldsymbol{i}}}{(x^2+y^2)^{3/2}} - \frac{x\,\widehat{\boldsymbol{j}}}{(x^2+y^2)^{3/2}} = \frac{\rho}{\rho^3}\left(\sin\phi\,\widehat{\boldsymbol{i}} - \cos\phi\,\widehat{\boldsymbol{j}}\right) = -\frac{\widehat{\boldsymbol{\phi}}}{\rho^2} \tag{3}$$

(b)

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(\frac{e^{-\rho^2}}{\rho}\right)\right] = -2e^{-\rho^2}, \tag{4}$$

$$\boldsymbol{\nabla}\cdot\mathbf{p} = \frac{1}{\rho}\frac{\partial(-1/\rho^2)}{\partial\phi} = 0. \tag{5}$$

(c) Since $\mathbf{F}$ and $\mathbf{p}$ are orthogonal (because $\widehat{\boldsymbol{\rho}}\perp\widehat{\boldsymbol{\phi}}$), the sin of the angle between the two vectors is 1 and you don't have to calculate it. Hence the vector product $\mathbf{F}\times\mathbf{p}$ is the product of the magnitudes of the two vectors, oriented in the third dimension (conventionally, $\widehat{\boldsymbol{\rho}}\times\widehat{\boldsymbol{\phi}} = \widehat{\boldsymbol{k}}$):

$$\mathbf{F}\times\mathbf{p} = \left(\frac{e^{-\rho^2}}{\rho}\right)\left(-\frac{1}{\rho^2}\right) = -\frac{e^{-\rho^2}}{\rho^3}\widehat{\boldsymbol{k}}. \tag{6}$$

3.

$$\begin{aligned}I &= 2\int_0^2 r\,dr\int_0^{\pi/2} d\phi\,\mathbf{r}\cdot\mathbf{r}(\mathbf{r}\cdot\mathbf{A})\\ I &= 2\int_0^2 r\,dr\int_0^{\pi/2} d\phi\,r^2(x+2y) = 2\int_0^2 r^4\,dr\int_0^{\pi/2}(\cos\phi + 2\sin\phi)\,d\phi\\ &= 2\times\frac{2^5}{5}\,(\sin\phi - 2\cos\phi)|_0^{\pi/2} = \frac{64}{5}(1-(-2)) = \frac{192}{5}.\end{aligned}$$