# PHYS2170 Mathematical Methods 4

Problems Class 5: Solutions

1. Recall the definition of the moment of inertia $I$ of a body with uniform mass density $\mu$ about an axis running through the body at point $\mathbf{r}_0$,

$$I = \mu \int dV \ R_\perp^2,$$

where $R_\perp$ is the perpendicular (shortest) distance from a point $\mathbf{r}$ in the bulk to axis of rotation.

In cylindrical coordinates aligned with the cylinder, $R_\perp^2 = \rho^2$. Since the mass density can be written as $\mu = M/(\pi R^2 L))$, we can write the moment of inertia in cylindrical polar coordinates as

$$I = \frac{M}{\pi R^2 L} \int_0^L dz \int_0^{2\pi} d\phi \int_0^R \rho \, d\rho \, \rho^2 = \frac{M}{\pi R^2 L} L \, 2\pi \, \frac{1}{4} R^4 = \frac{1}{2} M R^2.$$

2. Consider the line integral

$$I = \int_C \left[ \frac{y \, dx}{(x^2+y^2)^{3/2}} - \frac{x \, dy}{(x^2+y^2)^{3/2}} \right],$$

where the curve $C$ is the semi-circle (of radius 3) from $(-3, 0)$ to $(3, 0)$.

(a) The vector $\mathbf{V}$ can be extracted $d\mathbf{r} = dx\widehat{\boldsymbol{i}} + dy\widehat{\boldsymbol{j}} = dr\,\widehat{\boldsymbol{r}} + r\,d\phi\widehat{\boldsymbol{\phi}}$ as

$$\mathbf{V} = \frac{y\widehat{\boldsymbol{i}}}{(x^2+y^2)^{3/2}} - \frac{x\,\widehat{\boldsymbol{j}}}{(x^2+y^2)^{3/2}} = \frac{r\sin\theta\widehat{\boldsymbol{i}}}{r^3} - \frac{r\cos\theta\widehat{\boldsymbol{j}}}{r^3} = -\frac{\widehat{\boldsymbol{\phi}}}{r^2}.$$

(b) The integral is then

$$I = \int_C \mathbf{V} \cdot d\mathbf{r} = \int_{\pi, r=3}^{0} (-\frac{\widehat{\boldsymbol{\phi}}}{r^2}) \cdot d\mathbf{r} = -\int_{\pi, r=3}^{0} \frac{1}{r^2} \, r \, d\phi = \frac{\pi}{3}.$$