PHYS2170 Mathematical Methods 4

Problems Class 6

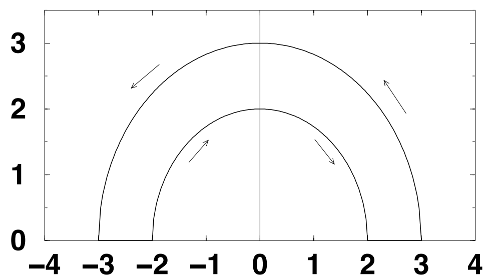
1. Evaluate the following surface integrals, where S is the curved surface of a cylinder of radius R and height from $z = 0$ to $z = H$:

   (a) $\mathbf{I}_1 = \int_S d\mathbf{S}$
   [The answer is a vector, and be careful...the unit vector $\widehat{\boldsymbol{\rho}}$ depends on the azimuthal angle ϕ, $\widehat{\boldsymbol{\rho}} = \cos\phi\widehat{\boldsymbol{i}} + \sin\phi\widehat{\boldsymbol{j}}$].

   (b) $J = \int_S \mathbf{V} \cdot d\mathbf{S}$, where

   (i) $$\mathbf{V} = \frac{x\,\widehat{\boldsymbol{i}} + y\,\widehat{\boldsymbol{j}}}{(x^2 + y^2)^{1/4}}$$ (ii) $$\mathbf{V} = y\,\widehat{\boldsymbol{i}} - x\,\widehat{\boldsymbol{j}}$$

2. Consider the line integral (in the $x-y$ plane), $\oint_C \mathbf{V}\cdot d\mathbf{r}$, where C is the contour sketched below and



$$\mathbf{V} = \frac{y\hat{\mathbf{i}}}{(x^2 + y^2)^{1/2}} - \frac{x\hat{\mathbf{j}}}{(x^2 + y^2)^{1/2}}.$$

   (a) We will use Green's (or Stokes') Theorem to convert the line integral into an integral over the area enclosed by the contour C. Considering cylindrical polar coordinates, what are the limits of integration for the area enclosed?

   (b) Write the area integral that must be evaluated. Calculate the curl in either cartesian or polar coordinates, and express the final result in polar coordinates. The expression for the curl in cylindrical polar coordinates is $\boldsymbol{\nabla}\times\mathbf{V} = \left[\frac{\partial V_\phi}{\partial z} - \frac{1}{\rho}\frac{\partial V_z}{\partial \phi}\right]\widehat{\boldsymbol{\rho}} + \left[\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right]\widehat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho V_\phi) - \frac{\partial V_\rho}{\partial \phi}\right]\widehat{\boldsymbol{k}}$.

   (c) Compute the integrals.

   (d) Now compute the line integral explicitly and verify Green's theorem. (recall that $d\rho = \widehat{\boldsymbol{\rho}}d\rho + \widehat{\boldsymbol{\phi}}\,\rho\,d\phi + \widehat{\boldsymbol{k}}\,dz$).