PHYS2170 Mathematical Methods 4

Problems Class 6: Solutions

1. (a) The area element is $d\mathbf{S} = \widehat{\boldsymbol{\rho}}\rho\, d\phi\, dz$. The cylindrical surface is defined by $\rho = R, \phi : 0 \to 2\pi, z : 0 \to H$. The unit vector ρ is a function of angle ϕ according to $\widehat{\boldsymbol{\rho}} = \cos\phi\,\widehat{\boldsymbol{i}} + \sin\phi\,\widehat{\boldsymbol{j}}$. So,

$$\mathbf{I}_1 = \int_S d\mathbf{S} = R\int_0^{2\pi} d\phi \int_0^H dz \left(\cos\phi\,\widehat{\boldsymbol{i}} + \sin\phi\,\widehat{\boldsymbol{j}}\right) = \mathbf{0},$$

because the angular integrals vanish. Alternatively, it's easy to convince yourself from looking at the integral that the answer is zero by symmetry (for each contribution from one side of the cylinder there is an equal and opposite contribution from the other side).

(b) i. In cylindrical coordinates we have $\mathbf{V} = \rho^{1/2}\widehat{\boldsymbol{\rho}}$. Hence,

$$\int \mathbf{V}\cdot d\mathbf{S} = R\int_0^{2\pi} d\phi \int_0^H dz\widehat{\boldsymbol{\rho}}\cdot\rho^{1/2}\widehat{\boldsymbol{\rho}} = 2\pi R^{3/2}H = 2\pi H R^{3/2}.$$

ii. In cylindrical coordinates we have $\mathbf{V} = -\rho\widehat{\boldsymbol{\phi}}$. Hence $\mathbf{V}\cdot d\mathbf{S} = 0$ and the integral is zero.

2. (a) The integral to be calculated is over the area between the two semi-circles, $R = 2, 3$, from $\phi = 0 \to \pi$.

(b) The integral is

$$\oint_C \mathbf{V}\cdot d\mathbf{r} = \int_S \underbrace{\rho\, d\rho\, d\phi\widehat{\boldsymbol{k}}}_{d\mathbf{S}} \cdot\boldsymbol{\nabla}\times\mathbf{V} = \int_2^3 \rho\, d\rho \int_0^\pi d\phi(\boldsymbol{\nabla}\times\mathbf{V})_z$$

In polar coordinates, $\mathbf{V} = \sin\phi\widehat{\boldsymbol{i}} - \cos\phi\widehat{\boldsymbol{j}} = -\widehat{\boldsymbol{\phi}}$. The only contribution to the curl is from the ϕ components, $\boldsymbol{\nabla}\times\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\phi)\widehat{\boldsymbol{k}} = -\frac{1}{\rho}\widehat{\boldsymbol{k}}$ (all other derivatives vanish).

(c) Hence the integral is

$$I = -\int_2^3 \rho\, d\rho \int_0^\pi d\phi\frac{1}{\rho} = -\pi(3-2) = -\pi.$$

(d) Since we know what $\mathbf{V}$ is, we can write the line integral:

$$\oint_C \mathbf{V}\cdot d\rho = \oint(-\widehat{\boldsymbol{\phi}})\cdot\left[\widehat{\boldsymbol{\rho}}d\rho + \widehat{\boldsymbol{\phi}}\,\rho\, d\phi\right] == -\oint \rho\, d\phi.$$

The line integral is along the two semicircles and two small straight segments. Along the straight segments, $d\phi = 0$ and there is no contribution from the integral. Hence, we have

$$\oint_C \mathbf{V}\cdot d\mathbf{r} = -\left[\int_{\rho=3,\phi=0,\pi}\rho\, d\phi + \int_{\rho=2,\phi=\pi,0}\rho\, d\phi\right] = -\left[3(\pi) + 2(-\pi)\right] = -\pi.$$