PHYS2170 Mathematical Methods 4

Problems Class 7

1. Consider the line integral

$$J = \int_C \frac{y\,dx - x\,dy}{x^2}, \qquad \text{where } C \text{ is the line } y = 3x - 5 \text{ from } x = 1 \text{ to } x = 3.$$

   (a) Write the integral in the form $\int \mathbf{V} \cdot d\mathbf{r}$ and identify the vector field $\mathbf{V}$. Write $\mathbf{V}$ in both Cartesian and polar coordinates.

   (b) Show that $\mathbf{V}$ is a conservative vector field.

   (c) Calculate the potential Ψ such that $\mathbf{V} = \boldsymbol{\nabla}\Psi$. [To do this, one knows that $\frac{\partial \Psi}{\partial x} = V_x$ and $\frac{\partial \Psi}{\partial y} = V_y$. Integrate these two equations respectively with respect to x and y and compare them to find $\Psi(x, y)$. Alternatively, you can try and guess the function Ψ!!]

   (d) Hence evaluate the integral J by integrating $d\Psi$.

2. (a) Evaluate

$$\int_S \mathbf{W} \cdot d\mathbf{S},$$

   where $\mathbf{W} = 3z^2\rho\hat{\boldsymbol{\rho}}$ (cylindrical coordinates), and S is the surface of a mailing tube (cylinder without the end caps) of radius 3 and length 6 centered at the origin, aligned with its major axis in the z direction. [Recall that the scalar area element in cylindrical coordinates, at a given radius, is $dS = \rho dz\, d\phi$. You'll need the normal vector to this surface to construct the vector area element $d\mathbf{S}$.]

   (b) Verify explicitly that the divergence theorem holds,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{W}\, dV = \oint_S \mathbf{W} \cdot d\mathbf{S},$$

   by performing the relevant volume integral.

3. (a) In two dimensional polar coordinates, the electrostatic potential Ψ obeys $\nabla^2\Psi = 0$, where the Laplacian ∇^2 is given by

$$\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Psi}{\partial \phi^2}.$$

   Show by direct integration that the radially-symmetric solution is given by $\Psi(\rho) = A \ln \rho + B$, where A and B are constants.

   (b) In three dimensional spherical coordinates the Laplacian is given by

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Psi}{\partial \phi^2}.$$

   Show by direct integration that the spherically symmetric solution for Laplace's equation $\nabla^2\Psi = 0$ in three dimensions is $\Psi(r) = A/r + B$.