PHYS2170 Mathematical Methods 4

Problems Class 7: Solutions

1. (a) We can identify

$$\mathbf{V} = \frac{y}{x^2}\widehat{\boldsymbol{i}} - \frac{1}{x}\widehat{\boldsymbol{j}} = \frac{\sin\phi\widehat{\boldsymbol{i}} - \cos\phi\widehat{\boldsymbol{j}}}{r\cos^2\phi} = -\frac{\widehat{\boldsymbol{\phi}}}{r\cos^2\phi}$$

(b) To show it's conservative, take the curl:

$$\boldsymbol{\nabla}\times\mathbf{V} = \frac{1}{x^2} - \frac{1}{x^2} = 0. \tag{1}$$

The curl vanishes, so $\mathbf{V}$ is conservative.

(c) To construct the potential, we need $V_x = \partial_x\Psi$ and $V_y = \partial_y\Psi$, for some potential $\Psi(x, y)$. So, we have:

$$\partial_x\Psi = \frac{y}{x^2} \Longrightarrow \Psi = -\frac{y}{x} + g(y) \qquad \text{(integrate } x\text{)}$$
$$\partial_y\Psi = -\frac{1}{x} \Longrightarrow \Psi = -\frac{y}{x} + h(x) \qquad \text{(integrate } y\text{)},$$

where $g(y)$ and $h(x)$ are arbitrary functions. For Ψ to be the same function obtained either way, these function must equal each other, and are thus equal to a constant. Hence, we must have $\Psi(x, y) = -y/x + const.$ Alternatively, you might have just guessed this function [and then shown that it works...].

(d) The integral is given by

$$\int \mathbf{V}\cdot d\mathbf{r} = \int_A^B d\Psi = \Psi(x=3, y=3x-5) - \Psi(x=1, y=3x-5) = \Psi(3,4) - \Psi(1,-2)$$
$$= -2 - \frac{4}{3} = -\frac{10}{3}.$$

2. (a) The normal vector for the vector area element is in the $\widehat{\boldsymbol{\rho}}$ direction, for the surface of the cylinder. Hence, $d\mathbf{S} = \rho\, d\phi\, dz\widehat{\boldsymbol{\rho}}$.

$$\int_S \mathbf{W}\cdot d\mathbf{S} = \int_S 3z^2\rho\widehat{\boldsymbol{\rho}}\cdot(\rho dz\, d\phi\widehat{\boldsymbol{\rho}}) = \int_0^{2\pi} d\phi \int_{-3}^{3} dz\,(3z^2\rho^2)\Big|_{\rho=3}$$
$$= 27(2\pi)\left[\frac{1}{3}z^3\right]_{-3}^{3} = 27(4\pi)9 = 972\pi.$$

(b) From the divergence theorem:

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{W}\, dV = \oint_S \mathbf{W}\cdot d\mathbf{S} = \int_{\text{curved surface},\widehat{\boldsymbol{\rho}}} \mathbf{W}\cdot d\mathbf{S} + \int_{\text{top/bottom surfaces},\pm\widehat{\boldsymbol{k}}} \mathbf{W}\cdot d\mathbf{S}$$

The surface integral must be done over the entire surface bounding the cylinder. However, because $\mathbf{W}$ is parallel to $\widehat{\boldsymbol{\rho}}$ and thus perpendicular to $\widehat{\boldsymbol{k}}$, there will be no contribution from the area integrals on the ends of the cylinder (since the surface normals here are $\pm\widehat{\boldsymbol{k}}$ on the top and bottom, respectively, which implies that $\mathbf{W} \cdot d\mathbf{S} = 0$ on these surfaces). So the volume integral should give us the same thing we found previously. The divergence of $\mathbf{V}$ is

$$\boldsymbol{\nabla} \cdot \mathbf{W} = \tfrac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho) = \tfrac{1}{\rho} \frac{\partial}{\partial \rho} (\rho\, 3z^2 \rho) = \tfrac{1}{\rho} (6z^2\, \rho) = 6z^2.$$

The volume integral is thus

$$\begin{aligned} \int_V \boldsymbol{\nabla} \cdot \mathbf{W}\, dV &= \int_{-3}^{3} dz \int_0^{2\pi} d\phi \int_0^3 \rho\, d\rho\, (6z^2) = 6 \int_{-3}^{3} z^2\, dz\, (2\pi)\, (\tfrac{1}{2} 3^2) \\ &= (54\pi) \left[\tfrac{1}{3} z^3\right]_{-3}^{3} = 54\pi\, 2\, \tfrac{1}{3}\, 27 = 972\pi. \end{aligned}$$

3. For a radially symmetric solution, there is *no* angular dependence; hence only the radial derivatives are non-zero in the Laplacian operator.

   (a) For a radially symmetric solution in two dimensions, we have

   $$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Psi}{\partial \rho} \right) = 0.$$

   Integrating once yields $\rho \frac{\partial \Psi}{\partial \rho} = A$, or $\frac{\partial \Psi}{\partial \rho} = A/\rho$. Integrating once more yields $\Psi = A \ln \rho + B$.

   (b) For a radially-symmetric solution in three dimensions, we have

   $$\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) = 0.$$

   Integrating once yields $r^2 \frac{\partial \Psi}{\partial r} = A$, or $\frac{\partial \Psi}{\partial \rho} = A/r^2$. Integrating once more yields $\Psi = -A/r + B$.