PHYS2170 Mathematical Methods 4

Problems Class 8: Solutions

1. *Solve the following differential equations:*

$$\frac{\partial^2 y}{\partial x^2} = -9\,y,$$

with boundary condition $y(0) = 5$ *and* $y'(0)=0$*.* The easiest way is to recognize the two solutions, $y_1(x) = A\sin 3x$ and $y_2(x) = B\cos 3x$, so the general solution is $y(x) = y_1(x) + y_2(x)$. Applying the boundary conditions:

$$y(0) = B = 5$$
$$y'(0) = 3\,A = 0$$

yields the general solution $y(x) = 5\cos 3x$.

2. *The temperature $T(t)$ of a body cooling by radiation (dead star, last night's dinner, etc) obeys the following differential equation:*

$$\frac{dT}{dt} = -K\,T^4.$$

Find the general solution to this differential equation for a given initial condition $T(0) = T_0$.

We can integrate directly:

$$-\frac{dT}{T^4} = K\,dt \quad \Longrightarrow \quad \frac{1}{3}\left[\frac{1}{T(t)^3} - \frac{1}{T(0)^3}\right] = K\,t \Longrightarrow \quad T(t) = \left[\frac{1}{\frac{1}{T_0^3} + 3Kt}\right]^{1/3}$$

3. (a) $\frac{d^2y}{dt^2} = 16y$

   i. General solution. Try $y = e^{\lambda t}$:

$$y = e^{\lambda t} \Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4$$
$$\therefore y(t) = Ae^{4t} + Be^{-4t} = \tilde{A}\sinh 4t + \tilde{B}\cosh 4t.$$

   ii. Basis functions are either (e^{4t}, e^{-4t}) or $(\sinh 4t, \cosh 4t)$.

   (b) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0$

   i. General solution. Try $y = e^{\lambda t}$:

$$y = e^{\lambda t} \Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3 \quad \text{(twice)}$$
$$\therefore y(t) = Ae^{3t} + B\,t\,e^{3t} \quad \text{(use form for repeated roots).}$$

   ii. Basis functions are (e^{3t}, te^{3t}).

   (c) $\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 6\frac{dy}{dt} = 0$

   i. General solution. Try $y = e^{\lambda t}$:

$$y = e^{\lambda t} \Rightarrow \lambda^3 + 5\lambda^2 + 6\lambda = 0 \Rightarrow \lambda(\lambda + 3)(\lambda + 2) = 0 \Rightarrow \lambda = 0, -2, -3$$
$$\therefore y(t) = A + Be^{-2t} + C\,e^{-3t}$$

   ii. Basis functions are $(1, e^{-2t}, e^{-3t})$.