PHYS2170 Mathematical Methods 4

Problems Class 9

1. Show explicitly that the following functions satisfy the partial differential equation $c\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}$:

   (a) $f(x,t) = \sinh(x+ct)$

   (b) $f(x,t) = g(x+ct)$, where g is any differentiable function.

   $f(x,t) = \sinh(x+ct) : \frac{\partial f}{\partial x} = \cosh(x+ct)$, $\frac{\partial f}{\partial t} = c\cosh(x+ct)$.

   $f(x,t) = g(x+ct) : \frac{\partial f}{\partial x} = g'$, $\frac{\partial f}{\partial t} = c\,g'$, where $g' \equiv dg(u)/du$, and we have applied the chain rule.

   Substituting these directly, we see that both functions $f(x,t)$ satisfy $c\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}$.

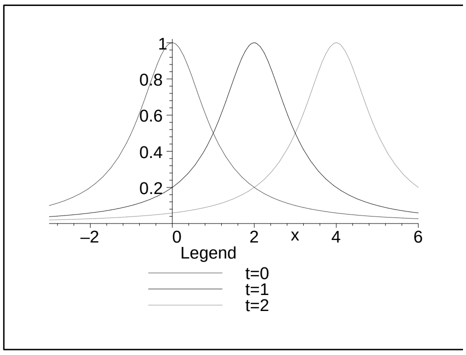
2. (a) $y(t) = A\sin 3t + B\cos 3t = A\,y_1(t) + B\,y_2(t)$. The solution space is 2 dimensional.

   (b) $y(t) = \sin(3t-4) = \sin 3t\cos 4 - \sin 4\cos 3t = \cos 4y_1(t) - \sin 4y_2(t)$

3. (a) Show that a solution to the partial differential equation

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 h(x-2t)}{\partial x^2} = h'' && (h' \equiv dh/dz)\\
\frac{\partial^2 f}{\partial t^2} &= \frac{\partial^2 h(x-2t)}{\partial t^2} = 4h'' && \text{(chain rule twice)}\\
\therefore 4\frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 f}{\partial t^2}.
\end{aligned}$$

   (b) For $h(z) = \frac{1}{1+z^2}$, we have:



   (c) Another solution is $f(x,t) = h(x+2t)$ (a pulse moving in the opposite direction).

4. Writing $p(x,t) = f(x-2t) + g(x+2t)$ gives the general solution. The wave speed must be $c = 2$ to satisfy the PDE. The boundary conditions are

$$p(x,0) = f(x) + g(x) = \begin{cases} 1 & (-0.1 < x < 0.1) \\ 0 & (|x| > 0.1) \end{cases}$$

$$\frac{\partial p}{\partial t}(x,0) = -2f'(x) + 2g'(x) = 0.$$

Integrating the second condition with respect to x, we have $f = g + A$, where A is a constant. Substituting into the first equation gives:

$$2g + A = \begin{cases} 1 & (-0.1 < x < 0.1) \\ 0 & (|x| > 0.1) \end{cases} \implies g = \begin{cases} (1-A)/2 & (-0.1 < x < 0.1) \\ -A/2 & (|x| > 0.1) \end{cases}$$

$$\therefore f = g + A = \begin{cases} (1+A)/2 & (-0.1 < x < 0.1) \\ +A/2 & (|x| > 0.1) \end{cases}$$

The constant A will cancel out when adding the functions together, so we can choose $A = 0$. Hence $p(x,t)$ is a sum of two pulses of height $1/2$. At time zero they are centered at the origin and add to give the initial condition, while at later times they have moved away from the origin.

Initial and later time ($t = 1$) solutions.