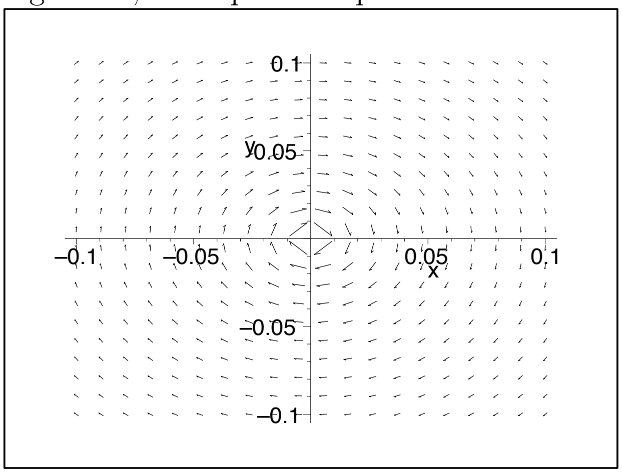
Assignment 1: Solutions. Number of points given in [brackets]. 29 points maximum.

1. The volume can be calculated using the scalar triple product, using the three vectors emanating from any vertex of the parallelpiped. For example, [3]

$$V = (\mathbf{v} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) = \mathbf{v} \cdot \mathbf{b} \times \mathbf{d} \qquad (\text{since} \quad \mathbf{a} = \mathbf{0})$$
$$= (1, -1, 1) \cdot (0, 0, 3) = 3.$$

2. (a) $\mathbf{V}(\mathbf{r}) = \dfrac{(y\hat{\mathbf{i}} - x\hat{\mathbf{j}})e^{-a(x^2+y^2)^{3/2}}}{x^2+y^2} = \dfrac{\rho\sin\phi\,\widehat{\boldsymbol{i}} - \rho\cos\phi\,\widehat{\boldsymbol{j}}}{\rho^2}e^{-a\rho^3} = -\dfrac{\widehat{\boldsymbol{\phi}}}{\rho}e^{-a\rho^3}.$ [3]

(b) To plot the vector *field* it is necessary to show how the vector field varies in both direction and magnitude, from point to point. This means drawing vectors all over the place! [2]



(c) In polars: to calculate $\boldsymbol{\nabla} \times \mathbf{V}$, note that there is only a ϕ component to $\mathbf{V}$; furthermore, this component only depends on ρ. Hence, we only need [looking up the identity for curl and extracting the relevant component]

$$\boldsymbol{\nabla} \times \mathbf{V} = \widehat{\boldsymbol{k}}\,\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\phi) = \widehat{\boldsymbol{k}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-e^{-a\rho^3}\right) = 3a\rho e^{-a\rho^3}\widehat{\boldsymbol{k}}.$$

In cartesians it's a slog. There's only a z component (To see this, note that the x and y components of $\mathbf{V}$ do not depend on z and that there is no z component of $\mathbf{V}$; and then stare at the usual determinant). The chain rule has to be applied many times. [4]

$$(\boldsymbol{\nabla} \times \mathbf{V})_z = \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} = \left\{\left[\frac{1}{x^2+y^2} - \frac{y(2y)}{(x^2+y^2)^2} + \frac{y}{x^2+y^2}(3a/2)(2y)(x^2+y^2)^{1/2}\right]\right.$$
$$\left. - (-)\left[\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} + \frac{x}{x^2+y^2}(3a/2)(2x)(x^2+y^2)^{1/2}\right]\right\} e^{-a(x^2+y^2)^{3/2}}$$
$$= \left[\frac{2(x^2+y^2) - 2x^2 - 2y^2}{(x^2+y^2)^2} + \frac{3a(x^2+y^2)(x^2+y^2)^{1/2}}{x^2+y^2}\right] e^{-a(x^2+y^2)^{3/2}}$$
$$= 3a\sqrt{x^2+y^2}\,e^{-a(x^2+y^2)^{3/2}}.$$

This is the same as the result in polar coordinates, since $\rho = \sqrt{x^2+y^2}$.

3. Given $h(x, y) = (x^2 + 1)/(1 + x^2 + y^3)$.

   (a) [3]

$$\boldsymbol{\nabla} h = \left(\widehat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \widehat{\boldsymbol{\jmath}}\frac{\partial}{\partial y}\right) h = \widehat{\boldsymbol{\imath}}\frac{2x\,(1 + x^2 + y^3) - (1 + x^2)(2x)}{(1 + x^2 + y^3)^2} + \widehat{\boldsymbol{\jmath}}\frac{-3y^2(1 + x^2)}{(1 + x^2 + y^3)^2}$$
$$= \frac{2x\,y^3\widehat{\boldsymbol{\imath}} - 3y^2(1 + x^2)\widehat{\boldsymbol{\jmath}}}{(1 + x^2 + y^3)^2}.$$

   (b) To find the slope in the direction of $-\widehat{\boldsymbol{\imath}} - \widehat{\boldsymbol{\jmath}}$, we need to project the gradient $\boldsymbol{\nabla} h$ along a unit vector in this direction. Hence, [2]

$$\text{slope} = -\frac{1}{\sqrt{2}}(\widehat{\boldsymbol{\imath}} + \widehat{\boldsymbol{\jmath}}) \cdot \boldsymbol{\nabla} h\bigg|_{x=1,y=2} = -1\frac{1}{\sqrt{2}}\frac{2x\,y^3 - 3y^2(1 + x^2)}{(1 + x^2 + y^3)^2}\bigg|_{x=1,y=2}$$
$$= \frac{1}{\sqrt{2}}\frac{-16 + 24}{(1 + 1 + 8)^2} = \frac{1}{\sqrt{2}}\frac{8}{100} = \frac{2}{25\sqrt{2}}.$$

   (c) The maximum slope at this point is the magnitude of the gradient at this point, or [2]

$$\sqrt{\boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} h} = \left|\frac{16}{100}\widehat{\boldsymbol{\imath}} - \frac{24}{100}\widehat{\boldsymbol{\jmath}}\right| = \frac{\sqrt{16^2 + 24^2}}{100} = \frac{2\sqrt{13}}{25}.$$

4. The Laplacian operator, in cartesian coordinates, is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. So: [4]

   (a) $\nabla^2(x^2 + 2xy + 3ze^{-(x+y)}) = [2 + 0 + 0] + [0 + 0 + 0] + \left[3ze^{-(x+y)} + 3ze^{-(x+y)} + 0\right] = 2 + 6ze^{-(x+y)}$.

   (b) $\nabla^2\left[e^{-5x}\sin 4y\cos 3z\right] = (25 - 16 - 9)e^{-5x}\sin 4y\cos 3z = 0$.

5. Maxwell's Equations for electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in free space are

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad (1)$$
$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \quad (2)$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3)$$
$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \quad (4)$$

   (a) Take the curl of Equation **??**:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} = \underbrace{\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2\mathbf{E}}_{\text{identity}} + \underbrace{\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \mathbf{B}}_{\text{change derivative order}} = 0$$

$$\underbrace{0 - \boldsymbol{\nabla}^2\mathbf{E}}_{\text{use } \boldsymbol{\nabla}\cdot\mathbf{E}=0} + \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}}_{\text{Use Eq. ??}} = 0 \qquad \Longrightarrow c^2\nabla^2\mathbf{E} = \frac{\partial^2}{\partial t^2}\mathbf{E}.$$

   This is the full wave equation in three dimensional space (rather than the one dimensional cousin for a wave on a string, $\frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial x^2}$). Note that this is actually *three* wave equations, one for each component of $\mathbf{E}$! A similar calculation can be peformed for $\mathbf{B}$. [3]

(b) If $\mathbf{B} = \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$ and $\mathbf{E} = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$, then Equation **??** implies:[3]

$$\nabla \times [\mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)] + \frac{1}{c} \frac{\partial [\mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)]}{\partial t} = 0$$

$$\underbrace{[\nabla \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)] \times \mathbf{E}_0}_{\mathbf{E}_0 \text{ constant}} + \frac{1}{c} \mathbf{B}_0 \frac{\partial}{\partial t} [\sin(\mathbf{k} \cdot \mathbf{r} - \omega t)] = 0$$

$$\left[\mathbf{k} \times \mathbf{E}_0 + \frac{1}{c}(-\omega)\mathbf{B}_0\right] \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = 0$$

This must be true for all points $\mathbf{r}$ and all times t; hence we must have $\mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0 / c$. That is, the magnetic and electric fields are polarized perpendicular to each other and, since we know that $\mathbf{k} \perp \mathbf{E}_0$ from Maxwell Equation (**??**), and $k = \omega / c$ [this can be found from the wave equation derived in the previous part], the electric and magnetic fields are equal in magnitude, in these units.