Please complete these questions, showing your working. Assignments should be handed in at the lecture or to my office (9.84). Total possible marks [30].

1. Compute $I = \int_S dA\, 2\, x^2$, where S is the region between a circle of **diameter** 6 centered at the origin and a square with **sides of length** 6 centered at the origin [**Hint:** Express the integral as a difference of two integrals] [5]

2. Consider the following line integral:

$$J = \int_C \mathbf{V} \cdot d\mathbf{r}, \qquad \mathbf{V} = y e^{xy} \hat{\mathbf{i}} + (x e^{xy} + 1)\, \hat{\mathbf{j}}$$

   where C is the curve from $x = 6$ to $x = 1$ along the line $y = x^3$ [7]

   (a) Evaluate this integral explicitly.

   (b) Show by taking the curl that $\mathbf{V}$ is conservative.

   (c) Integrate $\mathbf{V}$ to find the corresponding potential ϕ, such that

$$V_x = \frac{\partial \phi}{\partial x}, \qquad V_y = \frac{\partial \phi}{\partial y}.$$

   (d) Calculate J by evaluating ϕ at the endpoints of C, hence obtaining the same result as in (a).

3. Use Stokes' Theorem to evaluate the following line integral:

$$I = \oint_C \mathbf{W} \cdot d\mathbf{r}, \qquad \mathbf{W} = y x^2\, \hat{\mathbf{i}} - \left(x y^2 + z\right) \hat{\mathbf{j}} + x y e^{-z^2} \hat{\mathbf{k}},$$

   where C is the **counterclockwise** path around a loop in the $x-y$ plane. Let the loop traverse 1/3 of a circle of radius 3 centered at the origin, in the direction from $(0,0,0)$ to $(3,0,0)$ to $(3\cos\frac{2\pi}{3}, 3\sin\frac{2\pi}{3}, 0)$ to $(0,0,0)$. [5]

4. Calculate the following surface integral both explicitly (over the surface area) and by the divergence theorem (over the volume):

$$I = \oint_S \{x\, dy\, dz + y\, dx\, dz + z\, dx\, dy\},$$

   where S is the surface of a sphere of radius 2 [**Hint:** Write the integrand in vector form, $\mathbf{W} \cdot \mathbf{dS}$, and convert to spherical coordinates]. [6]

5. Gauss's Law for the electric field is:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho,$$

where ρ is the charge per unit volume at a given point. [7]

(a) Integrate this equation over an arbitrary volume and use the divergence theorem to relate a surface integral of the electric field $\mathbf{E}$ to the total charge Q enclosed inside a volume.

(b) Now we'll apply the divergence theorem (Gauss's Law) to calculate the electric field and potential around a wire. Assume a wire of infinite length aligned in the $\hat{\mathbf{z}}$ direction, with charge per unit *length* λ. What direction does the electric field point? [Use cylindrical coordinates (ρ, z, ϕ)]

(c) Assuming the electric field points in this direction with a magnitude E_0, perform the surface integral you obtained in (a), where the surface is a cylinder of radius R and length L encircling the wire, and hence find E_0 as a function of λ and R [You'll have to express the total charge enclosed in terms of λ]. **Draw a picture!**

(d) The electrostatic potential ψ is given by

$$\mathbf{E} = -\boldsymbol{\nabla}\psi.$$

Integrate the electric field you found in (c) from $\rho = \infty$ to $\rho = R$, assuming $\psi(\infty) = 0$, and hence find the electrostatic potential around a wire.

$$\left\{\text{recall that} \quad \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}.\right\}$$