Assignment 2: Solutions. Number of points given in [brackets]. 30 points maximum.

1. We compute the integral over the square, and subtract the integral over the circle. [5]

$$I_\Box = \int_{-3}^{3} dx \int_{-3}^{3} dy\, 2x^2 = 2\left(2\frac{1}{3}3^3\right)(6) = 216$$
$$I_\circ = \int_0^3 r\,dr \int_0^{2\pi} d\theta\left(2r^2\cos^2\theta\right) = 2\left(\frac{1}{4}3^4\right)\pi = \frac{81\pi}{2}$$
$$\Rightarrow I = I_\Box - I_\circ = 216 - \frac{81\pi}{2}.$$

2. *Consider the following line integral:* [7]

$$J = \int_C \mathbf{V}\cdot d\mathbf{r}, \qquad \mathbf{V} = ye^{xy}\hat{\mathbf{i}} + (xe^{xy}+1)\hat{\mathbf{j}}$$

where C is the curve from $x=6$ to $x=1$ along the line $y=x^3$;

(a) *Evaluate this integral explicitly.* To evaluate this, we replace $y=x^3$, $dy = 3x^2\,dx$, to find [(2)]

$$\begin{aligned} J &= \int_C \left[ye^{xy}\,dx + (xe^{xy}+1)\,dy\right] && \text{(take scalar product)}\\ &= \int_6^1 \left[x^3e^{x^4}\,dx + \left(xe^{x^4}+1\right)3x^2\,dx\right] && \text{(eliminate } y\text{)}\\ &= \int_6^1 \left[4x^3e^{x^4} + 3x^2\right]\,dx \\ &= \left[e^{x^4}+x^3\right]\Big|_6^1 = e - e^{1296} + 1 - 6^3 = e - 215 - e^{1296}. \end{aligned}$$

(b) *Show by taking the curl that $\mathbf{V}$ is conservative.* [(1)]

$$\nabla\times\mathbf{V} = \widehat{\boldsymbol{i}}\,(0-0) + \widehat{\boldsymbol{j}}\,(0-0) + \widehat{\boldsymbol{k}}\,(e^{xy} + xye^{xy} - xye^{xy} - e^{xy}) = \mathbf{0}$$

(c) *Integrate $\mathbf{V}$ to find the corresponding potential ϕ, such that* [(2)]

$$V_x = \frac{\partial\phi}{\partial x}, \qquad V_y = \frac{\partial\phi}{\partial y}.$$

We need to integrate,

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= ye^{xy} && \Longrightarrow \phi = e^{xy} + g_1(y) && \text{(integrate } x \text{ at fixed y)}\\ \frac{\partial\phi}{\partial y} &= xe^{xy}+1 && \Longrightarrow \phi = e^{xy} + y + g_2(x) && \text{(integrate } y \text{ at fixed x),} \end{aligned}$$

where $g_1(y)$ and $g_2(x)$ are, respectively, the constants of integration with respect to x and y. Since we must obtain the same function ϕ, we must have $g_1(y) = y + g_2(x)$. So, we can take $g_2(x)$ to be a constant (zero), while $g_1(y) = y$. Hence we can write the potential ϕ as

$$\phi = e^{xy} + y.$$

(d) *Calculate J by evaluating ϕ at the endpoints of C, hence obtaining the same result as in (a).* The result for ϕ is, of course, the same result we found above in (2b) (if we identify $y = x^3$ in the result). Hence the integral is, as it should be, exactly as we found it before. [(2)]

3. By Stokes' Theorem, [5]

$$
\begin{aligned}
I &= \oint_C \mathbf{W}\cdot d\mathbf{r} = \int_S d\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{W}) \\
&= \int_S \left(r\,dr\,d\phi\,\widehat{\boldsymbol{k}}\right)\cdot\boldsymbol{\nabla}\times\mathbf{W} && (d\mathbf{S}\text{ in polars}) \\
&= \int_S r\,dr\,d\phi\,(\boldsymbol{\nabla}\times\mathbf{W})_z = \int_S r\,dr\,d\phi\left(-y^2-x^2\right) && (z\text{ component of curl}) \\
&= -\int_S r\,dr\,d\phi\,r^2 = -\int_0^3 r^3\,dr\int_0^{2\pi/3}d\phi = -\frac{1}{4}3^4\frac{2\pi}{3} = -\frac{27\pi}{2}.
\end{aligned}
$$

The integral is taken with radius $r = 3$, and spans an angle of $2\pi/3$. Because the path is taken counterclockwise, we need a surface normal of $\widehat{\boldsymbol{k}}$ (right hand rule).

4. [6]

$$
\begin{aligned}
I &= \int_S \{x\,dy\,dz + y\,dx\,dz + z\,dx\,dy\} \\
&= \int_S \left(x\,\widehat{\boldsymbol{\imath}} + y\,\widehat{\boldsymbol{\jmath}} + z\,\widehat{\boldsymbol{k}}\right)\cdot \underbrace{\left(\widehat{\boldsymbol{\imath}}dy\,dz + \widehat{\boldsymbol{\jmath}}\,dx\,dz + \widehat{\boldsymbol{k}}\,dx\,dy\right)}_{\text{general vector area element } d\mathbf{A} \text{ in cartesian coordinates}} \\
&= \int_S \mathbf{r}\cdot d\mathbf{S} \qquad \text{(identifying the total vector area element)} \\
&= \int_S \mathbf{r}\cdot\hat{\mathbf{r}}(r^2\sin\theta d\theta d\phi) \\
&= r^3\int_S \sin\theta d\theta d\phi = 4\pi r^3 = 32\pi.
\end{aligned}
$$
[(3)]

Using the divergence theorem,

$$
\begin{aligned}
I &= \int_V \boldsymbol{\nabla}\cdot\mathbf{r}\,dV = \int_V 3\,dV \qquad \text{(divergence theorem)} \\
&= 3\int_V dV = 3\frac{4}{3}\pi 2^3 = 32\pi.
\end{aligned}
$$
[(3)]

5. Gauss's Law for the electric field is: [7]

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho,$$

where ρ is the charge per unit volume at a given point.

(a) Integrating over the an arbitrary volume gives [(1)]

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{E}\,dV = \int_V \rho\,dV \qquad \Rightarrow \qquad \oint_S \mathbf{E}\cdot d\mathbf{S} = Q.$$

[The integral of the charge density, charge per unit volume, over the whole volume gives the total enclosed charge Q.

(b) *Assume a wire of infinite length aligned in the* $\hat{\mathbf{z}}$ *direction, with charge per unit* length λ. *What direction does the electric field point? [Use cylindrical coordinates* (ρ, z, ϕ)*].* Since the system is radially symmetric, there can be no angular dependence to $\mathbf{E}$. Hence there is no component in the $\widehat{\boldsymbol{\phi}}$ direction, where we use cylindrical coordinates. Because the system is infinite in the z direction, there can be no dependence on z. By symmetry, there can be no difference in looking at the wire "right side up" or "upside down". That is, the solution must be symmetric under $z \to -z$. Hence there is no component in the $\widehat{\boldsymbol{z}}$ direction (since this component would change sign if the wire was flipped upside down. That leaves a component in the radial direction, parallel to $\widehat{\boldsymbol{\rho}}$, that can only depend on ρ. [(2)]

(c) Now we apply this to a cylinder around the wire: [(2)]

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \lambda Z,$$

where S is the surface of a cylinder of height Z and radius R, and λZ is the enclosed charge. Integrating:

$$\lambda Z = \oint \mathbf{E} \cdot d\mathbf{S} = \int_S dz \rho \, d\phi \widehat{\boldsymbol{\rho}} \cdot E_0 \widehat{\boldsymbol{\rho}} = \int_0^Z dz \int_0^{2\pi} d\phi \, R \, E_0 = 2\pi Z R E_0$$

$$\implies E_0 = \frac{\lambda}{2\pi R}$$

(d) The electrostatic potential ψ is given by $\mathbf{E} = -\boldsymbol{\nabla}\psi$. Hence, [(2)]

$$\int_\infty^{\mathbf{R}} \mathbf{E} \cdot d\mathbf{r} = -\int_\infty^{\mathbf{R}} \boldsymbol{\nabla}\psi \cdot d\mathbf{r} = -\psi(\mathbf{R}) + \psi(\infty).$$

Integrating the electric field yields

$$-\psi(\mathbf{R}) + \psi(\infty) = \int_\infty^R \frac{\lambda}{2\pi\rho} \widehat{\boldsymbol{\rho}} \cdot \widehat{\boldsymbol{\rho}} d\rho = \frac{\lambda}{2\pi} \ln \rho |_\infty^R = \frac{\lambda}{2\pi} \left(\ln\infty - \ln R \right).$$

The potential at ∞ is arbitrary, since what matters is always the relative potential. Hence we can ignore this infinite constant. Actually, there is no such thing as an infinite wire; hence far enough away compared to the true length of the wire the potential will decay as $1/r$ instead of $\ln r$, and the potential at infinity will vanish.