1. [2]

$$f(x) = \frac{1}{1-x} \simeq \left[\underbrace{\frac{1}{1-x_0}}_{f(x_0)} + \underbrace{\frac{1}{(1-x_0)^2}}_{f'(x_0)}(x-x_0) + \frac{1}{2!}\underbrace{\frac{2}{(1-x_0)^3}}_{f''(x_0)}(x-x_0)^2 + \dots \right]_{x_0=0}$$
$$= 1 + x + x^2 + \dots$$

2. Solve $dy/dx = -9y$ with boundary condition $y(1) = 12$. [2]

$$y = Ae^{-9x}; \qquad y(1) = 12 = Ae^{-9} \qquad \Longrightarrow \underline{y(x) = 12\, e^{-9(x-1)}}.$$

3. Solve $d^2y/dx^2 = -9y$ with boundary conditions $y(0) = 0$ and $y'(0) = 12$. [2]

$$y = A\sin 3x + B\cos 3x; \quad y(0) = B = 0; \quad y'(0) = 3A = 12 \quad \Longrightarrow \underline{y = 4\sin 3x.}$$

4. *Consider $d^4y/dx^4 + 4d^3y/dx^3 + 4d^2y/dx^2 = 0$.. Calculate the general solution $y(x)$. What is the dimension of the solution space?* [4]

Letting $y = e^{\lambda x}$ yields,

$$\lambda^4 + 4\lambda^3 + 4\lambda^2 = 0 \qquad \Longrightarrow \lambda^2(\lambda+2)^2 = 0.$$

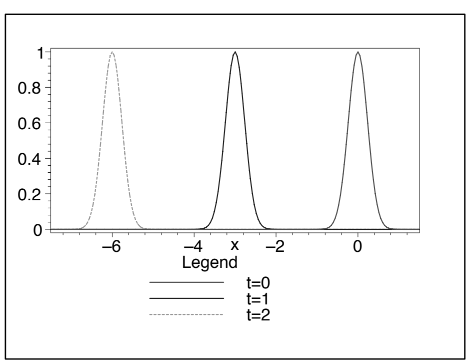
So, $\lambda = 0$ and $\lambda = -2$ are both two-fold repeated roots. The general solution is then the sum of four characteristic basis functions:

$$y(x) = \underbrace{a_1 + a_2\, x}_{\text{for } \lambda=0} + \underbrace{a_3 e^{-2x} + a_4\, x\, e^{-2x}}_{\text{for } \lambda=-2}.$$

The dimension of the solution space is 4.

5. *Solve $3\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}$ for $f(x,t)$, with boundary condition $f(x,0) = e^{-x^2}$. Sketch the solution for $t = 0, 1, 2$.* [4]

We showed in class that the general solution to this equation is $f(x,t) = g(x + 3t)$, where $g(z)$ is any function [Show this by substitution]. This is a wave pulse of some shape, moving to the left with speed $v = -3$. Hence, if $f(x, t = 0) = e^{-x^2}$, then the general solution is $\underline{f(x,t) = e^{-(x+3t)^2}}$. The solution at different times is then obtained by substituting $t = 0, 1, 2$:



6. (a) *Show by substitution that the functions* $\sin(x - ct)$ *and* $\sin(x + ct)$ *are solutions to* $c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$. [2]

Substituting, we find

$$-c^2 \sin^2(x \pm ct) = -c^2 \sin^2(x \pm ct), \qquad \text{as expected.}$$

(b) *Hence show that* $h(x,t) = \sin(x - ct) + \sin(x + ct)$ *is a solution.* [2]

This is the principle of linear superposition. Each function individually satisfies the wave equation. Upon substituting the sum into the PDE, the order of sum and partial derivatives can be interchanged to show that, if f_1 and f_2 satisfy the wave equation with speed c, then the sum does:

$$\begin{aligned} c^2 \frac{\partial^2 (f_1 + f_2)}{\partial x^2} &= \frac{\partial^2 (f_1 + f_2)}{\partial t^2}. \\ c^2 \frac{\partial^2 f_1}{\partial x^2} + c^2 \frac{\partial^2 f_2}{\partial x^2} &= \frac{\partial^2 f_1}{\partial t^2} + \frac{\partial^2 f_2}{\partial t^2}. \\ c^2 \frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_1}{\partial t^2} &= \frac{\partial^2 f_2}{\partial t^2} - c^2 \frac{\partial^2 f_2}{\partial x^2} \\ 0 &= 0 \end{aligned}$$

(c) [2]

$$\begin{aligned} h(x,t) &= \sin(x - ct) + \sin(x + ct) \\ &= \sin x \cos ct - \cos x \sin ct + \sin x \cos ct + \cos x \sin ct = 2 \sin x \cos ct. \end{aligned}$$

The solution is a sine wave in the x direction that oscillates up and down in time: a standing wave.