PHYS2170: Maths 4 (PDE section)

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Contents

1 Ordinary Differential Equations

1.1 Introduction

The *order* of an ODE is the highest derivative, while the *degree* of the ODE is the highest power of the dependent function f. For example, the following is a third order ODE of second degree:

$$\frac{dy}{dx}\,y + 6\frac{d^3y}{dx^3} = \sin 3x. \tag{1}$$

The first term has two powers of the dependent function y, and the second term has three derivatives, hence third order. This is an *inhomogeneous* ODE, because we have an additional function of x (on the right hand side).

This simplest ODE is the first order linear (first degree) homogeneous ODE,

$$\frac{df}{dx} = \alpha f(x), \qquad \text{or} \qquad \frac{df}{dx} - \alpha f(x) = 0. \tag{2}$$

This is *homogeneous* because every term in linear in f or linear in some derivative of f. This can be solved immediately (by assuming $f = e^{\lambda x}$, substituting into (2), and solving to find $\lambda = \alpha$), and the general solution is

$$f = A\,e^{\lambda x}. \tag{3}$$

A is the *constant of integration*, and we need a *boundary condition* to find the specific solution.

• **Generally, an n^{th} order ODE requires the specification of n boundary conditions, one for each derivative that must be integrated and gives rise to a constant of integration.**

The boundary condition picks out which function of the class of solutions is desired, and the ODE "propagates" the information on the boundary throughout the whole space.

The ODE above can also be written more generally as:

$$\frac{df}{dx} - \alpha f = 0 \tag{4a}$$

$$\left(\frac{d}{dx} - \alpha\right) f = 0 \tag{4b}$$

$$\mathcal{L} f = 0, \tag{4c}$$

where we have defined the *linear differential operator*

$$\mathcal{L} \equiv \frac{d}{dx} - \alpha. \tag{5}$$

This notation is very useful, for we can thus write ODE's very compactly. For example, we can write the following ODE

$$\frac{d^2 f}{dx^2} - 6\frac{df}{dx} + 12f = \sin x \tag{6}$$

as

$$\mathcal{L}f = \sin x, \tag{7}$$

if we define the linear operator (of second order) to be

$$\mathcal{L} \equiv \frac{d^2}{dx^2} - 6\frac{d}{dx} + 12. \tag{8}$$

1.2 Forced first order ODEs

An inhomogeneous differential equation has a "forcing term", *e.g.*the function $f(t)$ below:

$$\frac{dy}{dt} + \alpha\, y = f(t). \tag{9}$$

The general solution to this equation is:

$$y(x) = y_C(x) + y_p(x), \tag{10}$$

where $y_C(x)$ is the complementary function, which satisfies the homogeneous ODE

$$\frac{dy_C}{dx} + \alpha y_C = 0, \tag{11}$$

and y_p is the *particular integral.* This approach applies for linear ODE's of *any* order:

- **The solution to *any* linear inhomogenous ODE**

$$\mathcal{L}y = f(t), \tag{12}$$

where $\mathcal{L}$ is a linear operator of any order, can be written as the sum of a complementary function and the particular integral.

For some ODE's (*e.g.*first order ODE's with constant coefficients, as above), the particular integral can be solved exactly. In other cases the typical strategy is to *guess* a solution of the form of the forcing function. Typical functions that show up are polynomials, exponential functions, and sinusoidal (sin and cos) functions. The general solution to a first order homogeneous ODE with constant coefficient α (Eq. 9) is

$$y(t) = y(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\, f(s)ds. \tag{13}$$

Note that in this case s is a dummy variable of integration. The complementary function decays with an exponential, while the forcing function gives impulses at all times t' before the current time, and these pulses decay with the same exponential function. All of these contributions are added together (integrated).

1.3 Second Order homogeneous ODE's

To solve a second order ODE, *e.g.*

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \tag{14}$$

we again (as almost always!!) assume $y = e^{\lambda t}$. Substitution gives $\lambda = \pm i\omega$, and thus the general solutions are $y = e^{\pm i\omega x}$. Since we are typically dealing with real functions it is easier to write the general solution in terms of sine and cosine functions:

$$y = A \sin \omega x + B \cos \omega x. \tag{15}$$

1. There are two derivatives in a second order ODE.
2. There are two constants of integration (here, A and B) in a second order ODE.
3. There are two linearly independent *eigenfunctions* (here, they are $\sin \omega x$ and $\cos \omega x$) that comprise the general solution to a second order ODE.

The *eigenfunctions* are often called *basis functions.* There is a very strong analogy between the basis functions of the solution space of an ODE and the *basis vectors* (for example, $\widehat{\boldsymbol{i}}, \widehat{\boldsymbol{j}}, \widehat{\boldsymbol{k}}$) of a vector space.

Example 1:

1. Show that the general solution to

$$\frac{dy}{dx} - k^2 y = 0 \tag{16}$$

is either*

$$y = Ae^{kx} + Be^{-kx} \qquad \text{or} \qquad y = C \sinh kx + D \cosh kx. \tag{17}$$

Note that here there are two equally valid sets of basis functions (either exponentials or hyperbolic functions). This is analogous to vectors, where we have many possible sets of basis vectors (cartesians, polars, etc) from which to choose.

Example 2: Consider the following ODE,

$$\frac{d^2y}{dt} + 2\frac{dy}{dt} + 5y = 0. \tag{18}$$

Assuming $y = e^{\lambda t}$ and substituting, we find the following *characteristic polynomial* $P(\lambda)$, which must equal zero:

$$P(\lambda) \equiv \lambda^2 + 2\lambda + 5 = 0. \tag{19}$$

Solving this quadratic equation yields two roots (as we *should* have, for a second order ODE),

$$y = -1 \pm 2i. \tag{20}$$

*Recall that $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

There are two eigenfunctions, of the form $y = e^{-t\pm 2it} = e^{t}e^{\pm 2it}$. Since we are typically considering y to be a real function, it is easier to express $e^{\pm 2it}$ in terms of sinusoidal functions, so that the general solution is

$$y(t) = e^{-t}\left(A\sin 2t + B\cos 2t\right). \tag{21}$$

Two boundary conditions are needed to specifiy A and B. For example, initial conditions $y(0) = 5$ and $y'(0) = 0$ leads to $y(t) = 5e^{-t}\cos 2t$.

Example 3: A general second order ODE with constant coefficients has the form

$$a\frac{d^2y}{dt} + b\frac{dy}{dt} + cy = 0, \tag{22}$$

where a, b, c are constants. Again we assume $y = e^{\lambda t}$ and upon substituting we find the characteristic equation,

$$P(\lambda) \equiv a\lambda^2 + b\lambda + c = 0. \tag{23}$$

The two roots can be solved by the quadratic equation, $\lambda = (-b \pm \sqrt{b^2 - 4ac})/2a$, and there are three possibilities:

1. $b^2 > 4ac$: two real roots λ_1, λ_2, and the characteristic equation factors into the form (after dividing through by a)

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0. \tag{24}$$

   The two eigenfunctions are $e^{\lambda_1 t}, e^{\lambda_2 t}$ and the general solution is $y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$.

2. $b^2 < 4ac$: two complex roots, which can be written as $\lambda = \alpha \pm i\beta$ with $\alpha = -b/(2a)$ and $\beta = \sqrt{b^2 - 4ac}/(2a)$. The eigenfunctions are of the form $e^{\alpha t}\, e^{\pm i\beta t}$, as we found above, or one can use $e^{\alpha t}\sin\beta t$ and $e^{\alpha t}\cos\beta t$ if desired. Hence, the general solution is $y = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$.

3. $b^2 = 4ac$: two identical roots! In this case the characteristic equation factors into the form

$$(\lambda - \lambda_0)(\lambda - \lambda_0) = 0, \qquad \text{where} \qquad \lambda_0 = -\frac{b}{2a}. \tag{25}$$

   The general eigenfunctions for repeated roots λ_0 are $e^{\lambda_0 t}, te^{\lambda_0 t}$, and the general solution is $y = Ae^{\lambda_0 t} + Bte^{\lambda_0 t}$.

1.4 Linear Superposition

Consider a linear homogeneous differential equation,

$$\mathcal{L}u = 0, \tag{26}$$

where $\mathcal{L}$ is a linear operator. This can be *any* linear operator, involving any number of derivatives and also including non-constant coefficients. If u_1 and u_2 are linearly independent solutions to the differential equation, *i.e.*

$$\mathcal{L}u_1 = 0, \qquad \mathcal{L}u_2 = 0, \tag{27}$$

then $a_1u_1 + a_2u_2$ is also a solution, where a_1 and a_2 are constants. This can be seen by substitution:

$$\begin{aligned}
\mathcal{L}(a_1u_1 + a_2u_2) &= \mathcal{L}(a_1u_1) + \mathcal{L}(a_2u_2) && \text{(28a)}\\
&= a_1\mathcal{L}u_1 + a_2\mathcal{L}u_2 \qquad (a_1, a_2\,\text{constants}) && \text{(28b)}\\
&= a_1(0) + a_2(0) = 0. && \text{(28c)}
\end{aligned}$$

Hence, once we have found *all* linearly independent solutions, the general solution is the superposition of all of these solutions. This is also true for partial differential equations, and will be very useful later.

1.5 General homogeneous ODE's with constant coefficients

Much of what we have learned above can be applied to homogeneous ODE's of any order (with constant coefficients). Consider the following ODE

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_1\frac{dy}{dx} + a_0y = 0. \tag{29}$$

We can also write this as

$$\mathcal{L}y = 0, \tag{30}$$

where the linear operator in this case is

$$\mathcal{L} \equiv a_n\frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \ldots + a_1\frac{d}{dx} + a_0 \tag{31}$$

Again we substitute $y = e^{\lambda x}$ into the ODE, and arrive at the characteristic equation

$$P(\lambda) \equiv a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0. \tag{32}$$

This is an n^{th} order polynomial equation, which has n roots. That is the equation can, in principle (for example, Maple and Mathematica can always generate these roots) be written in the form

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_n) = 0, \tag{33}$$

where the n roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvalues are generally complex (note that complex numbers will always come in pairs), and the general solution is given by the superposition of n independent eigenfunctions:

$$y(x) = \sum_{i=1}^{n} A_iy_i(x). \tag{34}$$

For each root λ_i that appears uniquely, the eigenfunctions are $y_i(x) = e^{\lambda_i x}$. If a root is repeated g times, then that root has g eigenfunctions, given by

$$y_i(x) = e^{\lambda_i x}, xe^{\lambda_i x}, \ldots, x^{g-1}e^{\lambda_i x}. \tag{35}$$

Example 4: Find the general solution to the following ODE for the shape of a string $y(x)$:

$$y'''' - 9y'' = 0. \tag{36}$$

Substituting $y = e^{\lambda x}$ we find the following characteristic equation:

$$\lambda^4 - 9\lambda^2 = 0. \tag{37}$$

This can be factored immediately to see the roots:

$$\lambda^2(\lambda^2 - 9) = \lambda^2(\lambda - 3)(\lambda + 3) = (\lambda - 0)^2(\lambda - 3)(\lambda + 3) = 0. \tag{38}$$

Hence, $\lambda = +3$ and $\lambda = -3$ appear as unique roots, while $\lambda = 0$ is repeated twice. The eigenfunctions for the repeated roots are e^{0x} and xe^{0x}, or just 1 and x, while the other eigenfunctions are $y = e^{\pm 3x}$. Thus, the general solution is given by

$$y(x) = A + Bx + Ce^{3x} + De^{-3x}. \tag{39}$$

2 Partial Differential Equations

2.1 PDE's: Introduction

Recall that ODE's took boundary conditions at a *point*, and propagated them along the space or time axis. PDE's describe functions of more than one variable, for example space and time, $u(x, t)$. By analogy with ODE's, we will see that PDE's propagate entire *functions* into a domain of time or space. For example, one can specify the initial *shape* of a string (at time $t = 0$) for all x, and then propagate this function (the shape) forward in time.

In this module we will only consider linear PDE's with constant coefficients. To see how a PDE arises naturally, consider the propagation of a wave pulse $h(x, t)$ moving at constant speed. In this case, the height at time $t + \delta t$ at point x is given by the height that was a bit farther back, at $x - v\delta t$, at time t:

$$h(x, \underbrace{t + \delta t}_{\text{now}}) = h(\underbrace{x - v\delta t}_{\text{here earlier}}, t). \tag{40}$$

Taylor expanding, we find

$$\frac{\partial h}{\partial t} = -v\frac{\partial h}{\partial x}, \qquad \text{or} \qquad \begin{cases} \dfrac{\partial h}{\partial t} + v\dfrac{\partial h}{\partial x} & = 0 \\ \left(\dfrac{\partial}{\partial t} + v\dfrac{\partial}{\partial x}\right) h & = 0 \end{cases} \tag{41}$$

This is a simple first order PDE that is satisfied by any wave pulse moving at constant speed. Notice that this can be satisfied by a pulse of *any* shape. By direct substitution you can verify *[Check!!]* that the general solution to this PDE is

$$h(x, t) = g(x - vt), \tag{42}$$

where $g(z)$ is *any* differentiable function. That is, one can reduce the function of two variables $h(x, t)$ to a function of a single variable, $g(z)$, as long as one makes the identification $z = x - vt$. This does not solve the PDE exactly, but restricts the class of solutions enormously.

Example 5: Solve the PDE

$$\frac{\partial f}{\partial t} + 3\frac{\partial f}{\partial x} = 0, \qquad \text{with} \qquad f(x,0) = e^{-3x^2}. \tag{43}$$

This is a travelling wave with speed $v = 3$, so the general solution is

$$f(x,t) = g(x - 3t). \tag{44}$$

At time zero we have $g(x - 3 * 0) = g(x) = e^{-3x^2}$. Hence the general solution at all times is given by restoring the time dependence to $g(x - 3t)$:

$$f(x,t) = g(x - 3t) = e^{-3(x-3t)^2}. \tag{45}$$

1. We found the class of solutions, $f(x,t) = g(x - vt)$. This is analogous to finding the functional form $y = Ae^{\lambda t}$ for a first order ODE $\frac{dy}{dt} = \lambda y$.
2. Then we applied the initial condition, which was a *function* of x at time zero, to find the specific function $g(z)$. This is analogous to applying an initial condition at a point, $y(0)$, to find the specific A for the ODE.

2.2 Main PDE's

1. **Wave Equation:** This arises in many areas of physics. For a wave on a string, specified by a height function $h(x,t)$, one can solve Newton's laws "$F = ma$" to find

   $$\underbrace{T\frac{\partial^2 h(x,t)}{\partial x^2}}_{\text{Force}} = \underbrace{\rho\frac{\partial^2 h(x,t)}{\partial t^2}}_{m\times a((per length))} . \tag{46}$$

   Here, T is the tension, $\partial^2 h/\partial x^2$ is the local curvature, and ρ is the mass per unit length. Rearranging, we have the familiar form for the wave equation,

   $$\frac{\partial^2 h(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 h(x,t)}{\partial t^2} \qquad \textbf{Wave Equation}, \tag{47}$$

   where the wave speed is given by $c = \sqrt{T/\rho}$. This arises in electrodynamics as well, for the electric and magnetic vector fields:

   $$\boldsymbol{\nabla}^2 \mathbf{E}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2}. \tag{48}$$

   This is in fact *three* wave equations, one for each component E_x, E_y, E_z of the electric field, and the single space derivative has been replace by the three dimensional Laplacian operator.

2. **Diffusion Equation** The local concentration of diffusing particles $c(x,t)$, or the temperature field $T(x,t)$ in an equilibrating heat conductor, is specified by the diffusion equation:

   $$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2} \qquad \textbf{Diffusion Equation}, \tag{49}$$

where D is the *diffusion coefficient*. In three space dimensions this would be written as

$$\frac{\partial c(\mathbf{r},t)}{\partial t} = D\boldsymbol{\nabla}^2 c(\mathbf{r},t). \tag{50}$$

3. **Laplace's Equation** In steady state, the concentration field above has no time derivative, and hence satisfies (in two dimensions) the equation

$$\frac{\partial^2 c(x,y)}{\partial x^2} + \frac{\partial^2 c(x,y)}{\partial y^2} = 0 \qquad \textbf{Laplace Equation}. \tag{51}$$

   This is also satisfied by the electrostatic potential in free space, $\phi(\mathbf{r},t)$, where now second derivatives with respect to x, y, z appear:

$$\boldsymbol{\nabla}^2\phi(\mathbf{r}) = 0. \tag{52}$$

   In this module we will only deal with differential equations for functions of two independent variables, such as $c(x,t), \phi(x,y), h(x,t)$; but in physical situations spatial variables typically come in three dimensions, and the same techniques that we will use generally apply.

4. Other common linear PDE's include:

$$\boldsymbol{\nabla}^2\phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \qquad \textbf{Poisson Equation} \tag{53a}$$

$$\frac{-\hbar^2}{2m}\boldsymbol{\nabla}^2\psi(\mathbf{r},t) = i\hbar\frac{\partial\psi}{\partial t} \qquad \textbf{Schrödinger Equation} \tag{53b}$$

2.3 Some general solutions

Before jumping into the technology of solving PDE's, let's look at some general solutions and features.

2.3.1 D'Alembert's Solution to the Wave Equation

The wave equation can be written as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{54}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{55}$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0. \tag{56}$$

Now we can *factor* the derivative operator to obtain two equivalent expressions for the wave equation:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0. \tag{57}$$

You can check this by explicitly expanding everything out, and recalling that it doesn't matter what order you take the derivatives in, *i.e.* $\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}$. Note that each term is in fact the linear operator for a first order PDE that is solved by a wave pulse of speed $\pm c$. So, the left hand version of the equation is satisfied if

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0, \tag{58}$$

which corresponds to a wave pulse travelling at speed c, with solution $u = f(x - ct)$. The right hand version of Eq. (57) is satisfied if

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0, \tag{59}$$

which corresponds to a wave pulse travelling at speed $-c$,with solution $u = g(x+ct)$. Hence the general solution to the wave equation is

$$u(x,t) = f(x - ct) + g(x + ct), \tag{60}$$

where $f(z)$ and $g(z)$ are arbitrary functions. The functions are undetermined, and we requires two boundary conditions (typically $u(x,0)$ and $\frac{\partial u(x,0)}{\partial t}$) to determine them.

The general procedure is thus:

(i) Write down the general solution $u = f(x - ct) + g(x + ct)$.

(ii) Substitute the general solution into the initial conditions. For example, if the the initial conditions are $u(x,0) = p_1(x)$ and $\frac{\partial u(x,0)}{\partial t} = p_2(x)$, then we can write:

$$f(x) + g(x) = p_1(x) \tag{61}$$
$$-cf'(x) + cg'(x) = p_2(x). \tag{62}$$

(iii) Solve for $f(x)$ and $g(x)$.

(iv) Restore the time into the solutions you have found, that is, let $f(x) \to f(x - ct)$ (substitute $x - ct$ for x) and let $g(x) \to g(x + ct)$ (substitute $x + ct$ for x) and hence write down the desired solution.

The wave equation is a *hyperbolic* differential equation (because of the similarity to the equation for a hyperbola, which is $x^2 - y^2$ =constant.

2.3.2 Laplace's Equation

The Laplace Equation looks like the wave equation, with an imaginary speed! Consider Laplace's equation for a function $u(x,y)$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{63}$$

This is essentially a wave equation with speed given by $c^2 = -1$. Hence the solution is a superposition of wave pulses travelling at "imaginary" speeds $c = \pm i$:

$$u(x, y) = f(x + iy) + g(x - iy), \tag{64}$$

where $f(z)$ and $g(z)$ are any functions of a complex variable $z = x + iy$. Notice that if we choose $g(z) = f^*(z) = \tilde{f}(x - iy)$, then

$$u(x, y) = 2\,\mathcal{R}e(f), \tag{65}$$

where $\mathcal{R}e(f)$ is the real part of f, satisfies Laplace's equation. Similarly, if we choose $g = -f^*$, we find that

$$u(x, y) = 2i\,\mathcal{I}m(f), \tag{66}$$

where $\mathcal{I}m(f)$ is the imaginary part of f, satisfies Laplace's equation. Hence, the real and imaginary parts of any function $f(z)$ of a complex number $z = x + iy$ satisfy Laplace's equation! This is a very useful trick to know at times!

Example 6: Find a solution to Laplace's equation if $u(x, 0) = e^{-2x^2}$. We can see that two possible solutions are

$$f_\pm = e^{-2(x \pm iy)^2}, \tag{67}$$

since these two solutions have "speed" $\pm i$, and setting $y = 0$ recovers the desired boundary condition (up to constants that must be determined by another boundary condition). Hence we could write the general solution as

$$u(x, y) = f_+(x, y) + f_-(x, y) = Ae^{-2(x+iy)^2} + Be^{-2(x-iy)^2} \tag{68}$$

$$= e^{-2(x^2-y^2)}\left[Ae^{2ixy} + Be^{-2ixy}\right] \tag{69}$$

$$= e^{-2(x^2-y^2)}\left[C\cos 2xy + D\sin 2xy\right]. \tag{70}$$

As usual, we can write exponential of imaginary argument as sin and cos functions, for convenience. Note that:

1. There are two constants, either A and B or C and D, determined by two boundary conditions, which are typically functions specified on the boundaries.

2. The general solution involves both exponentials and sinusoidal functions. This is typical of solutions to Laplace's equation.

3. Laplace's equation is an *elliptic* PDE: recall the equation for an ellipse, $\frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} = C$, where C is a constant.

2.3.3 Diffusion equation

You can easily verify *[Check yourself!!]* that a solution to the diffusion equation (not the only one or a general one!!) is

$$c(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)}. \tag{71}$$

This is most easily verified by direct substitution into the diffusion equation,

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}. \tag{72}$$

This is the solution which obeys the initial condition

$$c(x, 0) = \delta(x), \tag{73}$$

where $\delta(x)$ is the Dirac delta function that is zero for all $x \neq 0$, and has a single spike at the origin. The result will be a lump of material at the origin (for example, sugar in solution) that gradually spreads out and flattens with increasing time. The diffusion coefficient D controls how fast the spreading out occurs.

The diffusion equation is a *parabolic* equation, because of the similarity to a parabola, which is described by the equation $y = x^2 + C$.

3 Solving PDE's: Separation of Variables

3.1 Main idea: Laplace's Equation as an example

The most common technique for solving PDE's is separation of variables This does not give *all* solutions, but is quite general and is usually the first line of attack. Consider solving Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{74}$$

The main steps are (the idea is the same for all PDE's):

1. Assume that the dependence on x and y factors, so that we can write

$$u(x, y) = f(x)g(y). \tag{75}$$

2. Substitute into the PDE. Noting that

$$\frac{\partial}{\partial x}[f(x)g(y)] = f'g, \tag{76}$$

where $f' \equiv \frac{\partial f}{\partial x}$, we can substitute Eq. (75) into Eq. (74) and then divide through by fg to obtain

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = \frac{f''}{f} + \frac{g''}{g} = 0, \tag{77}$$

where f'' and g'' imply taking two derivatives of the functions f and g with respect to their arguments, x and y respectively.

3. Since f''/f is a function of x and g''/g is a function of y, and this relation must hold for *all* x and y, f and g must satisfy the following coupled ODE's:

$$(I) \quad \begin{aligned} f'' &= -k^2 f \\ g'' &= +k^2 g \end{aligned} \qquad \text{or} \qquad (II) \quad \begin{aligned} f'' &= +k^2 f \\ g'' &= -k^2 g, \end{aligned} \tag{78}$$

where k^2 is called the *separation constant*. Which sign of k^2 is taken depends on the situation. The general solution is thus

$$(I)\ \begin{array}{ll} f(x) & = A\sin kx + B\cos kx \\ g(y) & = Ce^{ky} + De^{-ky} \end{array} \qquad (II)\ \begin{array}{ll} f(x) & = Ae^{kx} + Be^{-kx} \\ g(y) & = C\sin ky + D\cos ky. \end{array} \tag{79}$$

4. Hence the general solution is given by

$$u(x,y) = \begin{cases} [A\sin kx + B\cos kx]\left[Ce^{ky} + De^{-ky}\right] & (I) \\ \left[Ae^{kx} + Be^{-kx}\right][C\sin ky + D\cos ky] & (II). \end{cases} \tag{80}$$

5. Once the general solution has been found, the boundary conditions can be applied to find the separation constant k, and the arbitrary constants A, B, C, D. Which solution is chosen (oscillatory in x and exponential in y, or vice versa) must be determined according to the boundary conditions.

Example 7: Solve $\nabla^2\phi = 0$ *in two dimensions, on a flat domain* $0 \le x \le 1, 0 \le y \le 1$, *with boundary conditions*

$$\phi(0,y) = \phi(1,y) = \phi(x,1) = 0, \qquad \phi(x,0) = \sin \pi x. \tag{81}$$

To solve, we note that the boundary condition is sinusoidal in x. Hence, we expect the solution to be sinusoidal in x and exponential in y, and thus of the form

$$\phi(x,y) = (A\sin kx + B\cos kx)(Ce^{ky} + De^{-ky}). \tag{82}$$

Applying $\phi(x,0) = \sin \pi x$ implies:

$$\phi(x,0) = (A\sin kx + B\cos kx)(C + D) = \sin \pi x. \tag{83}$$

This implies that $B = 0$ and $k = \pi$. Moreover, we can now set $A = 1$ without losing generality (or just absorb the definition of A into C and D. Upon setting $A = 1$ we thus have

$$\phi(x,0) = \sin \pi x(C + D) = \sin \pi x \qquad \Rightarrow \qquad \underline{C + D = 1}. \tag{84}$$

Now notice that the two boundary conditions $\phi(0,y) = \phi(1,y) = 0$ are automatically satisfied because of the sinusoidal function $\sin \pi x$. This leaves us with the final boundary condition, $\phi(x,1) = 0$. Upon substituting the general solution into this boundary condition we find:

$$\phi(x,1) = \sin \pi x(Ce^{\pi} + De^{-\pi}) = 0 \qquad \Rightarrow \qquad \underline{Ce^{\pi} + De^{-\pi} = 0} \tag{85}$$

The boundary conditions thus give two simultaneous equations, Eq. (84,85), that must be solved to find the coefficients C and D. Solving, we find

$$C = -\frac{e^{-\pi}}{2\sinh \pi}, \qquad D = \frac{e^{\pi}}{2\sinh \pi}, \tag{86}$$

and upon substituting and rearranging *[Check yourself!!]*, the general solution can be written as

$$\phi(x,y) = \frac{\sin \pi x \sinh \pi(1-y)}{\sinh \pi}. \tag{87}$$

3.2 Wave Equation: general solution

We can solve the wave and diffusion equations in the same way. For the wave equation, for example

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2}. \tag{88}$$

We assume a solution of the form $h(x,t) = f(x)g(t)$. Substituting, we find

$$\frac{f''}{f} = \frac{1}{c^2}\frac{g''}{g} = -k^2, \tag{89}$$

where as before, the only solution is when f''/f and g''/g are both constants. In this case we choose a negative separation constant, although there are some (rare) physical situations where a positive separation constant is physically sensible. The solutions are sinusoidal, given by

$$f(x) = A\sin kx + B\cos kx, \qquad g(t) = C\sin ckt + D\cos ckt. \tag{90}$$

The general solution is thus

$$h(x,t) = (A\sin kx + B\cos kx)(C\sin ckt + D\cos ckt). \tag{91}$$

As before, we choose k, A, B, C, D according to the boundary conditions (or, in this case, typically one is given initial conditions in time).

Example 8: Find the most general solution to the wave equation for a string held at a separation of length L, whose endpoints are held fixed, i.e. $h(0,t) = h(L,t) = 0$. The solution is sinusoidal functions in x and t, of the form

$$h = (A\sin kx + B\cos kx)(C\sin ckt + D\cos ckt). \tag{92}$$

The boundary conditons are:

$$h(0,t) = B(C\sin ckt + D\cos ckt) = 0 \tag{93}$$

$$h(L,t) = (A\sin kL + B\cos kL)(C\sin ckt + D\cos ckt) = 0. \tag{94}$$

From the first condition we must have $B = 0$. Hence, the second

$$h(L,t) = A\sin kL(C\sin ckt + D\cos ckt) = 0. \tag{95}$$

We may take $A = 1$ as before. Thus, for this boundary condition to be satisfied at all times we must have $\sin kL = 0$, which is equivalent to the condition

$$kL = n\pi \qquad \Longrightarrow \qquad k = \frac{n\pi}{L}, n = 1, 2, 3, ... \tag{96}$$

Hence, any k parametrized by all integers n yields a possible solution. These solutions are sinusoidal functions of different period, which are linearly independent. By linear superposition, we may add all of these together to find the most general solution:

$$h(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right). \tag{97}$$

Example 9: Find the solution to the fixed end condition above, with initial condition $h(x,0) = 2sin\pi x/L$ and $\frac{\partial h(x,0)}{\partial t} = 0$.

From the general solution above we see that there is only one k value, corresponding to $n = 1$ or $k = \pi/L$. Hence the general form is just

$$h(x,t) = \sin\frac{\pi x}{L}(A\cos\frac{c\pi t}{L} + B\sin\frac{c\pi t}{L}). \tag{98}$$

The initial conditions are

$$h(x,0) = \sin\frac{\pi x}{L}(A * 1 + B * 0) = 2\sin\frac{\pi x}{L} \tag{99}$$

$$\frac{\partial h(x,0)}{\partial t} = c\,\sin\frac{\pi x}{L}(A * 0 - B * 1) = 0 \tag{100}$$

$$\tag{101}$$

Hence, $A = 2, B = 0$, and the solution is

$$h(x,t) = 2\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L}. \tag{102}$$

For more complicated initial conditions, we will need to write the initial conditions as a Fourier series, and then we can resolve all the coeffiecients A_n and B_n. This will be the subject of the final section.

3.3 Diffusion: general solution

Now we can solve the diffusion equation in the same way. Begin with the diffusion equation for the concentration $c(x,t)$ of a diffusing species in solution, for example:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}. \tag{103}$$

Separating variables and letting $c(x,t) = f(x)g(t)$, we find

$$\frac{1}{D}\frac{g'}{g} = \frac{f''}{f} = -k^2, \tag{104}$$

where we have chosen a negative separation constant†. The general solution is thus

$$c(x,t) = e^{-Dk^2t}(A\sin kx + B\cos kx), \tag{105}$$

where, as usual, k, A, B are determined by the boundary and initial conditions. As before, we can always write a general solution as a superposition of solutions for different k (or n, in the case where k is parametrize by an integer):

$$c(x,t) = \sum_k e^{-Dk^2t}(A_k\sin kx + B_k\cos kx), \tag{106}$$

where separate A_k and B_k must be determined for each k.

†A postive separation constant will yield solutions that *grow* exponentially in time; these are usually, but not always, unphysical solutions.

4 Solving PDE's: Fourier Series Methods

This is a powerful technique, and used, in practice, all the time. In fact, when you perform a scattering experiment [perhaps the most important experimental probe in physics] you are actually performing a Fourier analysis!

4.1 Fourier Series

The Fourier Series for a function $f(x)$ with period L:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right) \tag{107}$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2n\pi x}{L}\right)dx \tag{108}$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2n\pi x}{L}\right)dx. \tag{109}$$

The inversion formula are derived by integrating each side of the Fourier Series by $\cos 2m\pi x/L$ and $\sin 2m\pi x/L$, and integrating:

$$\int f(x)\cos\frac{2m\pi x}{L}\,dx = \int \cos\frac{2m\pi x}{L}\,dx\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right)\right] \tag{110}$$

The, we use the following:

$$\int_{-L/2}^{L/2} dx\ \cos\frac{2m\pi x}{L}\,dx\,a_0 = 0 \tag{111}$$

$$\int_{-L/2}^{L/2} dx\ \cos\frac{2m\pi x}{L}\,dx\sin\left(\frac{2n\pi x}{L}\right) = 0 \qquad \text{even * odd function} \tag{112}$$

$$\int_{-L/2}^{L/2} dx\ \cos\frac{2m\pi x}{L}\,dx\cos\left(\frac{2n\pi x}{L}\right) = \begin{cases} 0 & (n \neq m) \\ L/2 & n = m. \end{cases} \tag{113}$$

Hence only one term survives from the sum in the original integral (the mth cosine term), and

$$\int f(x)\cos\frac{2m\pi x}{L}\,dx = \frac{L}{2}a_m. \tag{114}$$

The same procedure can be performed for the sine coefficients, b_n

Notes:

1. Make use of even and odd functions!!

$$\text{even function} \qquad x, x^3, \sin 3x, \ldots \qquad f(x) = f(-x) \tag{115}$$

$$\text{odd function} \qquad 3, x^2, \cos 4x, \ldots \qquad f(x) = -f(-x). \tag{116}$$

Table 1: Solutions to the fundamental PDE's. In all cases the most general solution is obtained by adding together (according to linear superposition) the solutions for all k, A, B that are allowable by the boundary conditions. For example, $h(x,t) = \sum_k h_k(x,t)$, where $k = n\pi/L, B_k = 0$, is the most general solution for a wave on a string with boundary conditions $h(0, y = h(L,y) = 0$. The choice of solutions to Laplace's equation depends on the boundary conditions.

Laplace	**Wave**	**Diffusion**
$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$	$\frac{\partial^2 h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2}$	$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$
$\phi_k(x,t)$	$h_k(x,t)$	$c_k(x,t)$
$(A\sin kx + B\cos kx)(Ce^{ky} + De^{-ky})$ or $(Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$	$(A\sin kx + B\cos kx)(C\sin ckt + D\cos ckt)$	$e^{-Dk^2t}(A\sin kx + B\cos kx)$

These have the properties

$$\int_{-L}^{L} (\text{even}) = 2\int_{0}^{L} (\text{even}) \tag{117}$$

$$\int_{-L}^{L} (\text{odd}) = 0 \tag{118}$$

$$(\text{even}) \times (\text{odd}) = (\text{odd}) \tag{119}$$

$$(\text{odd}) \times (\text{odd}) = (\text{even}) \tag{120}$$

$$(\text{even}) \times (\text{even}) = (\text{even}). \tag{121}$$

Hence, Fourier series for even functions $(6, x^2, \cosh(x), \ldots)$ contain only cosine function, Fourier series for odd functions $(x, \tanh x, \ldots)$ contain only sine functions. Functions that are neither even nor odd $(e^{-x}, (x+3x^2-6)^2, \ldots)$ contain *both* sines and cosines.

2. The integration is over a period, and can be split up any way that is convenient:

$$\int_0^L, \int_{-L/2}^{L/2}, \int_{.1}^{.1+L}, \ldots . \tag{122}$$

3. Note the analogy to vector spaces. *E.g.* for an even function $f(x)$ and a vector $\mathbf{V}$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos k_n x \qquad \mathbf{V} = \sum_{i=1}^{3} a_i \widehat{\boldsymbol{e}}_i \quad \widehat{\boldsymbol{e}}_i = (\widehat{\boldsymbol{i}}, \widehat{\boldsymbol{j}}, \widehat{\boldsymbol{k}}) \tag{123}$$

$$\cos k_n x \qquad \widehat{\boldsymbol{e}}_i \tag{124}$$

$$a_n \qquad a_i \tag{125}$$

$$\int f(x) \cos k_m x = \tfrac{2}{L} a_m \qquad \mathbf{V} \cdot \widehat{\boldsymbol{e}}_j = a_j \tag{126}$$

$$\text{infinite dimensions} \qquad \text{3 dimensions} \tag{127}$$

Example – Square Wave As an example, let's calculate the Fourier representation of a square wave, specified by

$$f(x) = \begin{cases} -1 & (-L/2 < x < 0) \\ 0 & (x < 0 < L/2). \end{cases} \tag{128}$$

Then, this function can be made periodic by just repeating it along the x-axis. The first trick to note is that $f(x)$ is an *odd* function. That is, it satisfies $f(-x) = -f(x)$. Hence, there are *no* cosine terms (they're even, so the coefficients a_n, including a_0, will vanish), and the series is just

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right). \tag{129}$$

The coefficients are given by

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2n\pi x}{L}\right)dx. \tag{130}$$

$$= 2\frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2n\pi x}{L}\right)dx, \tag{131}$$

where to get to the second line I have used the fact that the *product* $f(x)\sin 2n\pi x/L$ is an *even* function, so I can just integrate over half of the period and multiply by two. You don't need to use this trick if you don't want; you can always omit that extra factor of two and integrate over the whole range. Now we substitute the function $f(x) = 1$ (which is true for $0 < x < L/2$), so that

$$b_n = \frac{4}{L}\int_{0}^{L/2} \sin\left(\frac{2n\pi x}{L}\right)dx = \frac{4}{L}\left(\frac{-L}{2n\pi}\right)\left[\cos\left(\frac{2n\pi x}{L}\right)\right]_{x=0}^{L/2} \tag{132}$$

$$= \frac{-4}{2n\pi}\left[\cos\left(\frac{2n\pi L/2}{L}\right) - \cos\left(\frac{2n\pi 0}{L}\right)\right] = \frac{-4}{2n\pi}(\cos n\pi - 1). \tag{133}$$

Now, for even n, we have $\cos n\pi - 1 = 1 - 1 = 0$. For odd n we have $\cos n\pi - 1 = -1 - 1 = -2$. Hence we have

$$b_n = \frac{4}{n\pi} \qquad (n\,\text{odd},\ i.e.\, n = 2m+1), \tag{134}$$

where m is any integer $m = 0, 1, 2, \ldots$ (this is just another way to write an odd number). The series can then be written as

$$f(x) = \begin{cases} \displaystyle\sum_{n\,\text{odd}} \frac{4}{n\pi}\sin\left(\frac{2n\pi x}{L}\right) \\ \displaystyle\sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi}\sin\left(\frac{2(2m+1)\pi x}{L}\right). \end{cases} \tag{135}$$

(either way of writing the solution is fine).

One common trick is to use Fourier series solutions to define infinite series. For example, if we choose $x = L/4$, we must have $f(x = L/4) = 1$. That is, we must have

$$f(L/4) = 1 = \sum_{n\,\text{odd}} \frac{4}{n\pi}\sin\left(\frac{2n\pi L/4}{L}\right) = \sum_{n\,\text{odd}} \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right). \tag{136}$$

Now, for $n = 1, 5, 9, \ldots$ we have $\sin n\pi/2 = 1$, while for $n = 3, 7, 11, \ldots$ we have $\sin n\pi/2 = -1$. Alternatively, let's write the sum in terms of $m = 1, 2, 3, \ldots$, so that

$$1 = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi}\sin\left(\frac{(2m+1)\pi}{2}\right) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi}(-1)^{m+1} \tag{137}$$

$$\pi = 4\sum_{m=0}^{\infty} \frac{1}{(2m+1)\pi}(-1)^{m}. \tag{138}$$

[To get the $(-1)^{m+1}$, note that $\sin(2m+1)\pi/2$ is $+1$ when $(2m+1)\pi/2 = \pi/2, 5\pi/2, 9\pi/2, \ldots$, that is, when $m = 0, 2, 4, \ldots$. Similarly, $\sin(2m+1)\pi/2$ is -1 when $m = 1, 3, 5, \ldots$]. Alternatively (and probably easier!!) you can write out the first four or five terms and "guess" the pattern.

4.2 Solving PDE's using Fourier Series

One of the most useful features is using Fourier series to solve PDE's. For example, consider the evolution of a wave on a string with speed c, with a shape at time zero given by

$$y(x,t=0) = \begin{cases} -1 & -\frac{L}{2} < x < 0 \\ 0 & < x < \frac{L}{2}, \end{cases}, \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \tag{139}$$

i.e. a square wave. Solutions to the wave equation have the form

$$y(x,t) = \begin{cases} \sin kx \cos \omega t \\ \cos kx \cos \omega t \\ \sin kx \sin \omega t \\ \cos kx \sin \omega t, \end{cases} \tag{140}$$

with $\omega = ck$ *[check this!]*.

The Fourier series for the initial condition is

$$y(x,t=0) = \sum_{nodd} \frac{4}{n\pi} \sin\left(\frac{2\pi n x}{L}\right) \tag{141}$$

This implies that, among the possible solutions we should keep solutions involving $\sin k_n x$ functions:

$$y(x,t) = \sum_{nodd} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin k_n x \qquad (k_n = 2\pi n/L). \tag{142}$$

Demanding $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ implies

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{nodd} \omega_n A_n \sin k_n x \qquad (k_n = 2\pi n/L) = 0. \tag{143}$$

Since *all the cosine and sine functions are independent functions*, we must have $A_n = 0$ for all n. Demanding the initial condition match the Fourier Series implies $B_n = 4/(n\pi)$, so that the general solution is

$$y(x,t) = \sum_{nodd} \frac{4}{n\pi} \cos \omega_n t \sin k_n x \qquad (k_n = 2\pi n/L). \tag{144}$$

This procedure can be used on the wave, Laplace, and diffusion equations:

1. Write the Fourier series for the initial condition $u(x,0)$ (wave or diffusion equation) or the shape along a boundaries $u(x,0)$ (Laplace's equation).

2. Using separation of variables, identify the particular combination of functions that are possible solutions to the PDE, given the Fourier Series. For example, if the Fourier series involves sine functions, then the solution is a linear superposition of

$$u_n(x,t) = \sin k_n x \times \begin{cases} A_n e^{-Dk_n^2 t} & \text{diffusion equation} \\ A_n \cos ck_n t + B_n \sin ck_n t & \text{Wave equation} \\ A_n \cosh k_n t + B_n \sinh k_n t & \text{Laplace's equation.} \end{cases} \tag{145}$$

3. Use the initial Fourier series, together with any other conditions, to calculate A_n and, if necessary, B_n, noting that the basis functions $\sin k_n x$ are *independent* functions, and the conditions must hold for them separately.