Single particle nonlocality with completely independent reference states

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Abstract. We describe a scheme to demonstrate the nonlocal properties of a single particle by showing a violation of Bell's inequality. The scheme is experimentally achievable as the only inputs are number states and mixed states, which serve as references to 'keep track of the experiment'. These reference states are created completely independently of one another and correlated only after all the measurement results have been recorded. This means that any observed nonlocality must solely be due to the single particle state. All the techniques used are equally applicable to massive bosonic particles as to photons and as such this scheme could be used to show the nonlocality of atoms.

Contents

1. Introduction

The issue of whether a single particle can exhibit nonlocality has inspired a great deal of discussion and debate in recent years. This controversy has been due, in no small part, to our adherence to the original model for testing locality as prescribed by Einstein, Podolsky, and Rosen (EPR) in their ground-breaking 1935 paper [1]. In their scheme, two separate observers each receive one particle from an entangled pair and measure some property of it. EPR argued that quantum mechanics predicted strange correlations in the measurement outcomes that provided compelling evidence that quantum mechanics was incomplete and reality had to be ascribed to some 'hidden variables' not contained in the formalism. In 1964, these speculations were put on a firm testable footing when John Bell derived an inequality that had to be obeyed by any local hidden variable theory [2]. This inequality was tested in a beautiful series of experiments, which showed that violations were indeed possible [3]–[7]. These results led to a reinterpretation of the paradox and the current prevailing view is no longer that quantum mechanics is incomplete, but rather that it cannot be a local realistic theory. That is, if we maintain the view that the physical world is real (i.e. it exists independent of our measurements), then we must abandon locality.

A violation of Bell's inequality in an EPR-type scheme has become the 'industry standard' for confirming the presence of nonlocality. However, such an approach runs into conceptual difficulties when we consider systems consisting of a single particle. If two observers share a single particle state, the detection of the particle at one location precludes the measurement of any property of the particle at the other location. This means that it is not possible to correlate the measurement outcomes and seek violations of Bell's inequality. Of course, if experiments cannot confirm the nonlocality of a single particle, then it is meaningless to attach any reality to it.

On the other hand, perhaps it would be more surprising if nonlocality, which is viewed as a *fundamental* feature of the physical world, were not evident for single particles but only appeared as an *emergent* property of two or more particles. It seems that if nonlocality is at the heart of quantum mechanics, it should not depend on how many particles (or excitations of the field) are present. The question then arises whether the problem of observing single particle nonlocality is not a fundamental one, but is simply due to the limitations of the schemes used for detecting it.

Tan, Walls and Collett (TWC) [8] were the first to propose a different way of detecting nonlocality that could be applied to the single particle case. Their scheme was later modified by Hardy [9] to extend the class of local models it ruled out. Both these proposals, however, required additional particles to be added to the system, which cast doubt on whether they truly demonstrated a *single* particle effect [10, 11]. Hardy responded by arguing that the predicted nonlocal properties vanished if the single particle was removed and so any observed nonlocality was directly attributable to this single particle state. A fertile debate ensued and a number of clever theoretical schemes [12]–[15] were put forward, however, many lacked a clear experimental route to implementation. This left them open to the criticism that they did not represent real testable effects.

One route that attracted a lot of attention involved converting single particle entanglement into a more recognizable two-particle entangled state [14, 15]. This involved passing a single photon through a $50:50$ beam splitter and ensuring that each output was incident on a single atom in its ground state. If the photon was carefully tuned to be resonant with an electronic

transition of the atom, the result would be an entanglement of two atoms in different states. Such an approach, however, was not without its critics [16, 17], who argued that it was all an unnecessary matter of semantics since the state could be rewritten in a first quantized picture which does not show entanglement.

Subsequent authors have proposed [18] and tested [19] protocols for quantum teleportation using only single particle states as the quantum resource. There is also a scheme [20] that makes use of mixed state inputs ensuring that only experimentally accessible states are required. A number of experiments have now also demonstrated violations of Bell's inequalities for single particles using a variety of techniques including cavity quantum electrodynamics (QED) [21] and tomographic detection methods [22]. One such experiment [16, 23] is based on the setup suggested by TWC and uses a clever combination of beam splitters and polarizers to ensure interferometric stability. While this experiment gives clear results, it is still subject to the possible 'loophole' that the two parties make use of shared reference states, i.e. the reference states have emerged as the two outputs from a single beam splitter. This process is problematic because it could lead to some additional nonlocality being introduced by the reference states. This would make it difficult to ascribe any observed nonlocality to the single particle state.

Evidently, much of the controversy associated with single particle systems stems from this introduction of additional particles. Unfortunately, they seem unavoidable if we are to keep track of the experiment. Instead, we need to ensure that they are added in such a way that they cannot contribute any nonlocality. This means that they must be generated by the two observers using only local operations and classical communication. In this paper, we show how this can be achieved in a manner that leaves no room for doubt since the two parties each employ a completely independently created mixed state as their reference. The scheme we present has the further advantage that it is equally applicable to massive particles as to photons and eliminates any need to violate number conserving superselection rules [9, 10]. Atoms are also likely to offer improved detection efficiencies over photons, which enhances their prospects.

2. Scheme 1: the basics

Let us begin by considering the simple scheme shown in figure 1. For definiteness, we shall describe the case of single photons, however, this scheme is equally applicable to massive *bosonic* particles. Bosons are required since we shall see that it is important that the particles involved can exist in coherent states. Here, we consider a vacuum state, $|0\rangle$, and a single photon state, $|1\rangle$, incident on the two input ports of a 50:50 beam splitter (BS1). After this beam splitter the state can be written in a Fock state basis as,

$$|\psi\rangle = |01\rangle_{12} + i|10\rangle_{12}, \qquad (1)$$

where the first term in each ket represents the number of particles in path 1 and the second, the number in path 2 (represented by the subscripts). For simplicity, we shall neglect normalization constants where they are unimportant.

As shown in figure 1, path 1 is sent to a second beam splitter (BS2) and path 2 to a third beam splitter (BS3), where they are combined with other states. The reference state on path 3 is taken to be a coherent state with a mean amplitude of 1 and phase ϕ_a, i.e. $|e^{i\phi_a}\rangle_c$ and the reference state on path 8 is taken to be a coherent state with a mean amplitude of 1 and phase ϕ_b, i.e. $|e^{i\phi_b}\rangle_c$, where the subscript c is used to distinguish a coherent state from a Fock state.
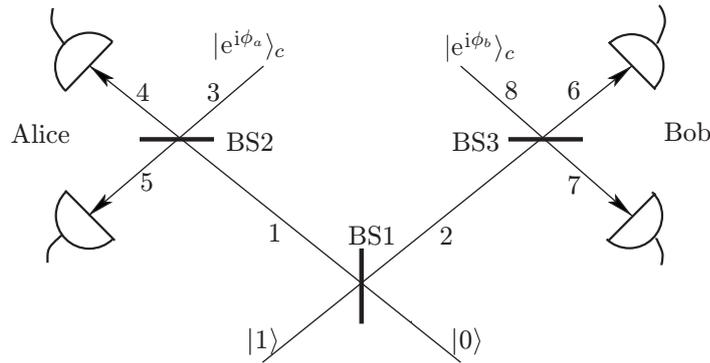
Figure 1. Experimental setup for scheme 1 which shows violations of the CHSH inequality for certain values of $\Delta\phi = \phi_b - \phi_a$. The reflectivities of BS2 and BS3 can be alternated between $\sin(\xi/2)$ and $\sin(\xi/2 + \pi/4)$, and $\sin(\eta/2)$ and $\sin(\eta/2 + \pi/4)$, respectively. The values of ξ and η are set so that $\xi - \eta = \pi/4$ to maximize the term $\sin(\xi - \eta) - \cos(\xi - \eta)$ in equation (13).

This means that the state of the system before entering BS2 and BS3 can be written as

$$|\psi\rangle = (|01\rangle_{12} + i|10\rangle_{12})(|0\rangle_3 + e^{i\phi_a}|1\rangle_3 + \cdots)(|0\rangle_8 + e^{i\phi_b}|1\rangle_8 + \cdots), \qquad (2)$$

where the latter two brackets show the input coherent states expanded in the Fock basis. To begin with, we will neglect all terms with more than one photon in these coherent states since we are only interested in single photon measurements. However, later in the paper we will consider what happens when we account for all the terms. This state is then transformed by beam splitters BS2 and BS3 and Alice and Bob record which of their detectors at the output ports registers a particle. Afterwards, by comparing their measurement outcomes, they seek nonclassical correlations that are a signature of nonlocality.

Importantly, BS2 and BS3 have variable reflectivities that Alice and Bob can rapidly switch. Straightforward schemes exist for implementing beam splitters for atoms [24, 25] with variable reflectivities, e.g. by rapid modulation of the trapping potential [26]. The role of these beam splitters with variable reflectivities can be understood by considering analogous schemes used to determine the EPR correlations between spin-1/2 particles. In such schemes, a spin singlet pair is produced: one particle being sent to Alice and the other to Bob. Alice and Bob can each choose to measure the spin of their particle in one of two spatial directions. They randomly choose which axis to measure in so as to eliminate the possibility of local hidden variables influencing the results. In our single particle scheme, instead of projecting onto different spatial directions, we project onto different superpositions of zero and one particles, i.e. different 'directions' when visualized on the Bloch sphere (see figure 2).

In analogy, with the spin singlet case, Alice and Bob each choose orthogonal measurement axes and these two sets of orthogonal axes are rotated with respect to each other. In figure 2, Alice's axes are shown at angles ξ and $\xi + \pi/2$ and Bob's are shown at η and $\eta + \pi/2$. The state represented by ξ, for example, is

$$|\psi\rangle = \cos(\xi/2)|0\rangle + e^{i\phi}\sin(\xi/2)|1\rangle. \qquad (3)$$

If we take the azimuthal angle, ϕ to be $\pi/2$, this is simply a state rotation which can be implemented by a beam splitter with reflectivity $\sin(\xi/2)$. The different detection axes
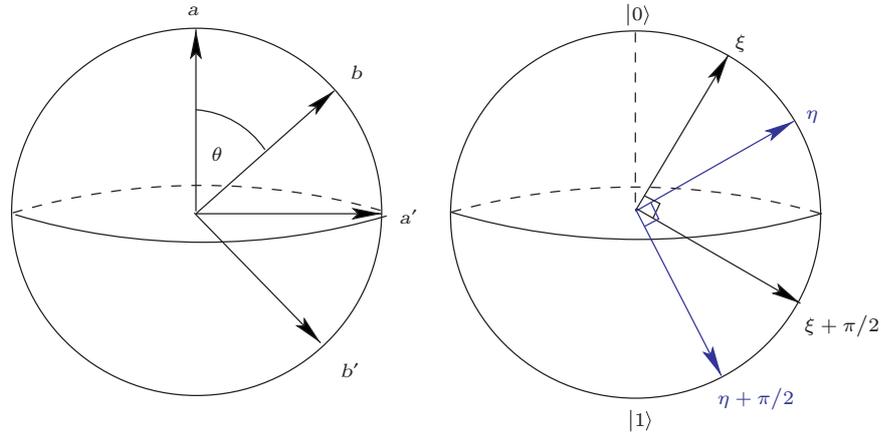
Figure 2. Left: when seeking violations of Bell's inequality for EPR pairs of spin-1/2 particles, Alice and Bob each choose two orthogonal spatial directions in which to make measurements. Alice's axes are labeled a and a' and Bob's are labeled b and b'. Right: the analogous measurement axes in our scheme for observing single particle entanglement correspond to different directions on the Bloch sphere. In this case, Alice's axes are in directions ξ and $\xi + \pi/2$ and Bob's are in directions η and $\eta + \pi/2$. The measurement axes can be adjusted by changing the reflectivities of beam splitters BS2 and BS3.

correspond to different reflectivities of BS2 and BS3, e.g. if Alice chooses to measure along axis ξ, she picks the reflectivity of BS2 to be $\sin(\xi/2)$ and if she chooses to measure along $\xi + \pi/2$, she picks the reflectivity to be $\sin(\xi/2 + \pi/4)$.

We would now like to see how (2) is transformed by the beam splitters. We are interested only in the cases where Alice and Bob each detect a single particle and we will consider first the case where Alice measures in direction ξ and Bob measures in direction η. The transform of BS2 on the possible inputs from paths 1 and 3 is therefore given by,

$$|10\rangle_{13} \rightarrow \cos(\xi/2)|10\rangle_{45} + i\sin(\xi/2)|01\rangle_{45}, \tag{4}$$

$$|01\rangle_{13} \rightarrow \cos(\xi/2)|01\rangle_{45} + i\sin(\xi/2)|10\rangle_{45}. \tag{5}$$

Using these transforms one can determine the probability of finding a particle in output paths 4 and 5. In a similar way (but replacing ξ with η) one can find the probability of finding a particle in the output paths 6 and 7. Combining these results it is possible to transform (2) to determine the probability of obtaining the state $|1010\rangle_{4567}$ (i.e. the probability of registering a click in the two upper detectors on paths 4 and 6, and nothing in the lower detectors on paths 5 and 7). This is given by,

$$P(\uparrow, \uparrow, \Delta\phi) = \frac{1}{2}\left[\sin^2\left(\frac{\xi + \eta}{2}\right)\cos^2\left(\frac{\Delta\phi + \pi/2}{2}\right) + \sin^2\left(\frac{\xi - \eta}{2}\right)\sin^2\left(\frac{\Delta\phi + \pi/2}{2}\right)\right], \tag{6}$$

where $\Delta\phi$ is the phase difference between the coherent states, i.e. $\phi_b - \phi_a$, and the two up-arrows denote that Alice and Bob, respectively, detect in their upper detectors. Using the same method, it is straightforward to show that $P(\downarrow, \downarrow, \Delta\phi) = P(\uparrow, \uparrow, \Delta\phi)$ and that the probabilities

for $|1001\rangle_{4567}$ and $|0110\rangle_{4567}$ are given by,

$$P(\uparrow, \downarrow, \Delta\phi) = P(\downarrow, \uparrow, \Delta\phi)$$
$$= \frac{1}{2}\left[\cos^2\left(\frac{\xi+\eta}{2}\right)\cos^2\left(\frac{\Delta\phi+\pi/2}{2}\right) + \cos^2\left(\frac{\xi-\eta}{2}\right)\sin^2\left(\frac{\Delta\phi+\pi/2}{2}\right)\right]. \quad (7)$$

We can also readily generate the corresponding probabilities for different choices of measurement axes, by simply replacing ξ with $\xi + \pi/2$ in (6) and (7), and/or replacing η with $\eta + \pi/2$.

We can now use the results in (6) and (7) to evaluate the Clauser, Horne, Shimony and Holt (CHSH) version of Bell's inequality [27], which has the form

$$S \equiv |E(\xi, \eta, \Delta\phi) + E(\xi + \pi/2, \eta, \Delta\phi) - E(\xi, \eta + \pi/2, \Delta\phi) + E(\xi + \pi/2, \eta + \pi/2, \Delta\phi)| \leqslant 2. \quad (8)$$

Any violation of this inequality can be interpreted as evidence of nonlocality in the system. The terms of the form $E(\xi, \eta, \Delta\phi)$ are expectation values or correlations between Alice and Bob's measurement outcomes given that Alice sets her beam splitter (and hence detection axis) to ξ and Bob sets his to η, for a given relative phase, $\Delta\phi$, between the reference states. We assign values of $\xi_\uparrow = +1$ and $\xi_\downarrow = -1$ to Alice's two possible measurement outcomes—detection in the upper and lower detectors, respectively. Similarly for Bob, we assign $\eta_\uparrow = +1$ and $\eta_\downarrow = -1$. The expectation values are then given by,

$$E(\xi, \eta, \Delta\phi) = \sum_{i,j\in\{\uparrow,\downarrow\}} \xi_i \eta_j P(i, j, \Delta\phi), \quad (9)$$

where $P(i, j, \Delta\phi)$ is the joint probability for different measurement outcomes for Alice and Bob. The general form of $P(i, j, \Delta\phi)$ for a *local* description of the system is,

$$P(i, j, \Delta\phi) = \int \rho(\lambda)\, P(i, \Delta\phi, \lambda) P(j, \Delta\phi, \lambda)\, d\lambda, \quad (10)$$

which attributes the correlations to some unspecified set of hidden parameters, λ with distribution $\rho(\lambda)$. If the joint probabilities have the form of (10), then it can be shown [28] that the CHSH inequality (8) must hold. Conversely, if this inequality is violated, a local description of the form of (10) is not valid. We would now like to see whether our scheme (as depicted in figure 1) violates the inequality.

The correlations, $E(\xi, \eta, \Delta\phi)$, can be calculated from (9) by substituting values from (6) and (7). This gives,

$$E(\xi, \eta, \Delta\phi) = +[P(\uparrow, \uparrow, \Delta\phi) + P(\downarrow, \downarrow, \Delta\phi)] - [P(\uparrow, \downarrow, \Delta\phi) + P(\downarrow, \uparrow, \Delta\phi)] \quad (11)$$

$$= -\sin^2(\Delta\phi/2 + \pi/4)\cos(\xi - \eta) - \cos^2(\Delta\phi/2 + \pi/4)\cos(\xi + \eta). \quad (12)$$

The other correlations can be found in a similar way,

$$E(\xi, \eta + \pi/2, \Delta\phi) = -\sin^2(\Delta\phi/2 + \pi/4)\sin(\xi - \eta) + \cos^2(\Delta\phi/2 + \pi/4)\sin(\xi + \eta),$$
$$E(\xi + \pi/2, \eta, \Delta\phi) = \sin^2(\Delta\phi/2 + \pi/4)\sin(\xi - \eta) + \cos^2(\Delta\phi/2 + \pi/4)\sin(\xi + \eta),$$
$$E(\xi + \pi/2, \eta + \pi/2, \Delta\phi) = -\sin^2(\Delta\phi/2 + \pi/4)\cos(\xi - \eta) + \cos^2(\Delta\phi/2 + \pi/4)\cos(\xi + \eta).$$

Substituting these expressions into (8), we obtain

$$S = \left| 2\sin^2\left(\frac{\Delta\phi + \pi/2}{2}\right)[\sin(\xi - \eta) - \cos(\xi - \eta)] \right|. \quad (13)$$

We see that this depends only on the relative orientation of Alice and Bob's measurement axes, i.e. $\xi - \eta$. This is what we would expect, since schemes that detect EPR correlations for pairs of entangled spin-1/2 particles depend only on the relative orientation of Alice and Bob's detectors.

In order to violate the CHSH inequality, S must have a value greater than 2. We would now like to find whether there are parameters that give rise to such a violation. S is maximized when each of the two factors in (13) have their maximum value. This occurs when $\Delta\phi = \pi/2$ and $\xi - \eta = 3\pi/4$. Substituting into (13) gives $S = 2\sqrt{2}$ which is greater than 2 and as such violates the CHSH inequality. This suggests that Alice and Bob should be able to arrange their detector orientations in such a way that they can observe evidence of single particle nonlocality.

3. Unfavorable measurement outcomes

So far we have only considered the case that Alice and Bob each detect a single particle and ignore all other 'unfavorable' measurement outcomes. This is problematic because we run into the well-known issue of subensemble post-selection [28], which calls into question whether our scheme can truly rule out local realistic models. In this section, we show that our scheme is still valid even when all the possible measurement outcome are accounted for. We follow an argument similar to one put forward by Popescu *et al* [29].

If we consider general coherent states, $||\alpha|e^{i\phi_a}\rangle_c$ and $||\alpha|e^{i\phi_b}\rangle_c$, on paths 3 and 8, respectively, of figure 1, the state of the system before entering BS2 and BS3 is,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{12} + i|10\rangle_{12})(||\alpha|e^{i\phi_a}\rangle_{c3})(||\alpha|e^{i\phi_b}\rangle_{c8}). \tag{14}$$

In section 2 above, we took $|\alpha| = 1$ and discarded all terms for which Alice and Bob did not each detect exactly one particle. Here, we shall keep $|\alpha|$ general and account for all the terms. State (14) can be rewritten in the form,

$$|\psi\rangle = \frac{|\alpha|}{\sqrt{2}}e^{-|\alpha|^2}\left[|01\rangle_{13}|10\rangle_{28} + i|10\rangle_{13}|01\rangle_{28}\right] + \sqrt{1 - |\alpha|^2 e^{-2|\alpha|^2}}|\Lambda\rangle, \tag{15}$$

where the square brackets contains the terms where Alice and Bob each receive precisely one particle (i.e. the terms that were retained in the analysis above) and $|\Lambda\rangle$ is all the remaining terms (i.e. the ones that were discarded above).

Alice and Bob now each have three possible measurement outcomes: they can detect a single particle in their upper detector, a single particle in their lower detector, or something else. For simplicity, we will assume perfect detector efficiencies throughout the analysis[1]. Following the prescription of Popescu *et al* [29], we assign a value of -1 to detection of a *single* particle in either lower detector or a value of $+1$ to any other result. It is straightforward to show that, with this assignment of values, the measurement outcomes of any system governed by a local model should obey the standard CHSH inequality, i.e. $S \leqslant 2$.

On the other hand, according to quantum mechanics,

$$E(\xi, \eta) = \langle\psi|A^\xi B^\eta|\psi\rangle, \tag{16}$$

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[1] In reality, imperfect detector efficiencies are a major source of concern. For example, with imperfect detectors it would not be possible to clearly distinguish good single particle events from imperfectly detected unfavorable events with more than one particle. The promise of improved detection efficiencies give atoms a distinct advantage over photons in such schemes.

where A^ξ is an operator (corresponding to Alice setting her beam splitter's reflectivity to $\sin(\xi/2)$), which has a nondegenerate eigenvalue of -1 corresponding to detection in the lower detector, and a multiply degenerate eigenvalue of $+1$ corresponding to the rest of Hilbert space, i.e. all other possible measurement outcomes. The other operator, B^η, is defined in a similar way. In other words, the operators A and B are equal to the usual detection operators on the subspace of 'favorable' cases (yielding -1 if the particle is detected in the lower detector and $+1$ for the upper detector) and equal to the identity operator on the subspace of 'unfavorable' cases.

We can now define $|\psi_1\rangle$ as the normalized projection of state (14) onto the subspace of favorable cases,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle_{13}|10\rangle_{28} + i|10\rangle_{13}|01\rangle_{28}\right], \tag{17}$$

where $|\Lambda\rangle$ is the corresponding projection onto the subspace of unfavorable cases. Now, since the local operators A and B do not mix these two cases, we can write,

$$S = \langle\psi|A^\xi B^\eta + A^{\xi+\pi/2}B^\eta - A^\xi B^{\eta+\pi/2} + A^{\xi+\pi/2}B^{\eta+\pi/2}|\psi\rangle$$

$$= |\alpha|^2 e^{-2|\alpha|^2}\langle\psi_1|A^\xi B^\eta + A^{\xi+\pi/2}B^\eta - A^\xi B^{\eta+\pi/2} + A^{\xi+\pi/2}B^{\eta+\pi/2}|\psi_1\rangle$$

$$+ (1 - |\alpha|^2 e^{-2|\alpha|^2})\langle\Lambda|A^\xi B^\eta + A^{\xi+\pi/2}B^\eta - A^\xi B^{\eta+\pi/2} + A^{\xi+\pi/2}B^{\eta+\pi/2}|\Lambda\rangle. \tag{18}$$

The expectation value on the bottom line is always 2 because the operators, A and B, acting on the unfavorable subspace always have eigenvalues of $+1$. The expectation value on the middle line is nothing other than the usual quantum CHSH expression computed in the $|\psi_1\rangle$ state and given by (13). Substituting these values into (18), we get

$$S = 2 + 2|\alpha|^2 e^{-2|\alpha|^2}\left[\sin^2\left(\frac{\Delta\phi + \pi/2}{2}\right)|\sin(\xi - \eta) - \cos(\xi - \eta)| - 1\right]. \tag{19}$$

For an ideal choice of parameters by Alice and Bob, i.e. $\Delta\phi = \pi/2$ and $\xi - \eta = 3\pi/4$, this expression reduces to

$$S = 2 + 2|\alpha|^2 e^{-2|\alpha|^2}\left(\sqrt{2} - 1\right) > 2. \tag{20}$$

S has its maximum value for $|\alpha| = 1/\sqrt{2}$, i.e. $S \approx 2.152$. This is sufficiently different from 2 to mean that experiments should be able to use this technique to rule out local realistic models by accounting for the results of *all* measurement outcomes.

A plot of S versus $|\alpha|$ is shown in figure 3. We see that S is only significantly greater than 2 for a narrow range of values of $|\alpha|$. It is important to the success of this scheme that a value is chosen in this range. In section 2, we used $|\alpha| = 1$, which is a suitable value, however, in the remainder of this paper we will use the best possible value, i.e. $|\alpha| = 1/\sqrt{2}$.

The discussion above shows how we can solve the problem of unfavorable measurement outcomes. However, there is another problem in so far as the scheme relies on the two reference coherent states having a fixed relative phase of $\Delta\phi = \pi/2$. If Alice and Bob each prepare their reference states separately, we know that there will be no fixed phase relationship between them. Furthermore, it is well known that the references should be described not as coherent states, but rather as mixed states of the form $\int_0^{2\pi}|\alpha e^{i\phi}\rangle\langle\alpha e^{i\phi}|d\phi$ [30]. This avoids problems with violating number conservation superselection rules and means that the two reference states have no absolute phase. If the two references are created separately, their relative phase will
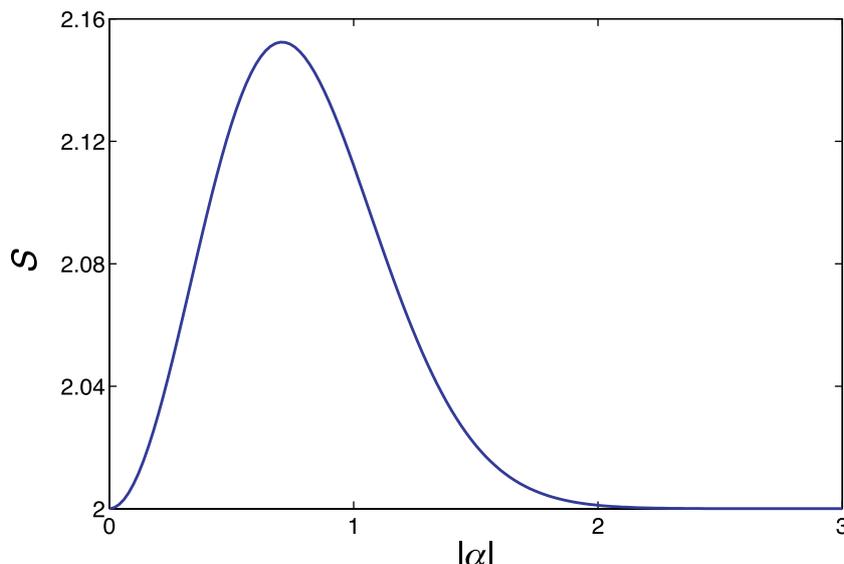
Figure 3. Plot of S as a function of $|\alpha|$ as given by equation (20). S differs significantly from 2 only over a small range of values of $|\alpha|$ and has a maximum value for $|\alpha| = 1/\sqrt{2}$.

vary randomly from shot to shot. This means that on an ensemble average—which is how these experiments must be performed—we need to average over all possible phase differences, $\Delta\phi$ in (19). When we do this for $|\alpha| = 1/\sqrt{2}$, we obtain,

$$S = 2 + \frac{1}{2e} \left[|\sin(\xi - \eta) - \cos(\xi - \eta)| - 2 \right], \tag{21}$$

which has a maximum value (when $\xi - \eta = 3\pi/4$) of $S = 2 - e^{-1}(1 - 1/\sqrt{2}) \approx 1.89$. Since this is less than 2, no violation of the CHSH inequality is possible in this case. In the following sections, we show how this problem can be overcome.

4. Scheme 2: fixing the phase

The key problem with the scheme outlined in the previous section is that it was not possible to fix the phase relationship between the two reference states. This problem can be overcome by a simple modification of the set up (see figure 4). This time, the two input coherent states in paths 3 and 8 are created by passing a coherent state, $|\beta\, e^{i\phi}\rangle_c$, and a vacuum state, $|0\rangle$ through a 50:50 beam splitter (BS4). The output from BS4 is $|i\beta\, e^{i\phi}/\sqrt{2}\rangle_{c3}|\beta\, e^{i\phi}/\sqrt{2}\rangle_{c8}$ meaning we have a fixed phase relationship of $\pi/2$ between the two input coherent states as required. By picking $\beta = 1$, we also obtain the correct amplitude in each state, i.e. $|ie^{i\phi}/\sqrt{2}\rangle_{c3}$ and $|e^{i\phi}/\sqrt{2}\rangle_{c8}$.

With these reference states, the total state just before BS2 and BS3 is (cf equation (14)),

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle_{12} + i|10\rangle_{12})(|ie^{i\phi}/\sqrt{2}\rangle_{c3})(|e^{i\phi}/\sqrt{2}\rangle_{c8}). \tag{22}$$

Calculating an expression for S in the same way as above gives,

$$S = 2 + 2|\alpha|^2 e^{-2|\alpha|^2} \left[|\sin(\xi - \eta) - \cos(\xi - \eta)| - 1 \right]. \tag{23}$$
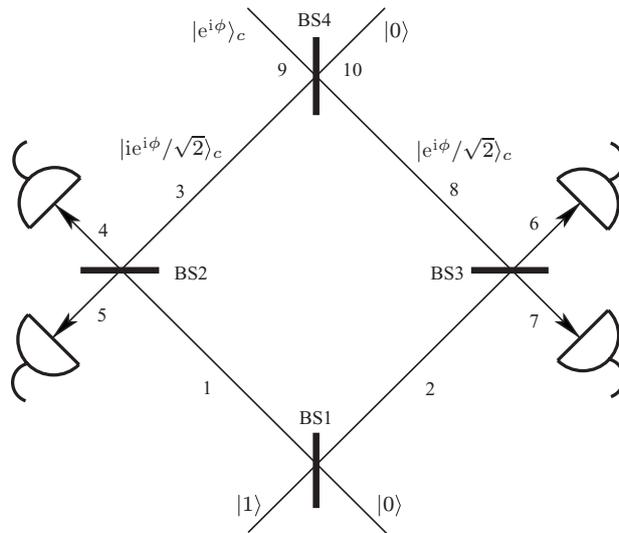
Figure 4. Experimental setup for scheme 2. The auxiliary state has a fixed phase relationship of $\Delta\phi = \pi/2$ and the reflectivities are chosen such that $\xi - \eta = \pi/4$.

This is exactly as we would expect from (19) since this new scheme fixes the value of the relative phase, $\Delta\phi = \pi/2$. Consequently this scheme can give rise to violations of the CHSH inequality since S depends only on $(\xi - \eta)$, the value of which can be controlled by the experimenters. Furthermore, the input on path 9 can be a mixed state of the form $\rho = \int |e^{i\phi}\rangle\langle e^{i\phi}| \, d\phi$, since the final result is independent of ϕ and so we are free to average over all values of ϕ without changing anything.

The coherent reference states, $|ie^{i\phi}/\sqrt{2}\rangle_{c3}$ and $|e^{i\phi}/\sqrt{2}\rangle_{c8}$, are important since they 'keep track of the experiment' and enable Alice and Bob each to make measurements. Since they are separable states, it can be argued that they do not introduce any additional nonlocality into the system and therefore the violation of the CHSH inequality is solely due to the single particle state on paths 1 and 2. The scheme presented in this section is closely related to the experiment of Hessmo *et al* [23] except that we have used beam splitters with variable reflectivity instead of polarizing beam splitters. This means that our scheme is readily applicable to atoms as well as photons.

5. Scheme 3: independent reference states

Although the scheme presented in the previous section should enable the nonlocality of a single particle to be observed, some readers may feel uneasy about how the reference states were created. Strictly speaking, in order for the reference states to not contribute any of the observed nonlocality, they should be created by only using local operations and classical communication. However, in scheme 2 they were created by a beam splitter, which performs a *global* operation on the two states.

For the special case that the input at path 9 is a coherent state (or a mixture of coherent states) and the input at 10 is a vacuum state, then this global operation does not pose any problems since we know that the outputs from BS4 will be completely separable. However, any deviation from this ideal case exposes the potential 'loophole' that some additional nonlocality
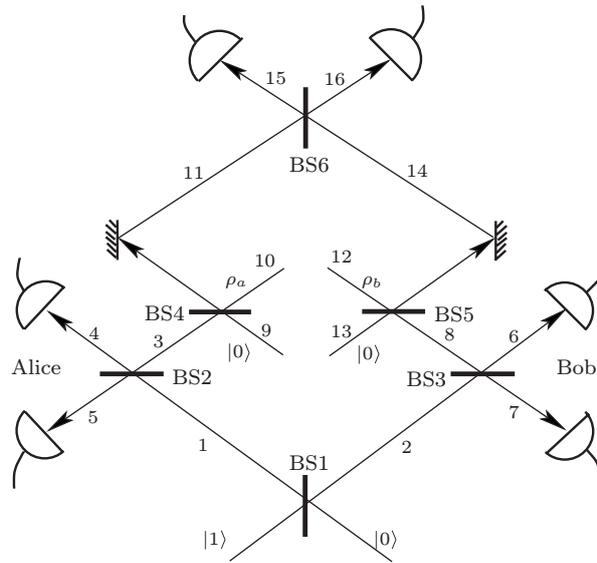
Figure 5. Experimental setup of scheme 3 which shows a violation of the CHSH inequality at certain values of $\Delta\phi$. The clicks at detectors 17 and 18 are used to determine $\Delta\phi$, therefore allowing a value of S to be determined. Mixed states are used as the auxiliary state and as such no superselection rules are violated.

is introduced by the reference states. This is a particularly pressing concern if we wish to apply this scheme to atoms. Atoms interact with one another through collisions and this leads to nonlinear terms in the Hamiltonian. These nonlinearities tend to number-squeeze the state [31, 32]. This means that, in general, the outputs from BS4 will not be separable and, as such, the reference states cannot be eliminated as the source of any nonlocality observed.

A solution to this problem is demonstrated in figure 5, whereby Alice and Bob each use a completely independent mixed state as their reference. Only after all the measurements have been taken are the relative phases of the reference states determined. This eliminates any possibility of nonlocality being due to these states. To begin with, we shall consider the case that Alice and Bob each use a mixed state of the form,

$$
\begin{aligned}
\rho_{a,b} &= \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |n\rangle \langle n| \\
&= \frac{1}{2\pi} \int_0^{2\pi} ||\alpha| \mathrm{e}^{\mathrm{i}\phi_{a,b}} \rangle \langle |\alpha| \mathrm{e}^{-\mathrm{i}\phi_{a,b}}| \mathrm{d}\phi_{a,b},
\end{aligned}
\tag{24}
$$

where the subscripts a and b denote the states belonging to Alice and Bob, respectively. This is a realistic choice for the reference states since it is the natural description of an atomic Bose–Einstein condensate. However, we shall see later that our scheme is not restricted by this choice. Equation (24) shows that the reference state can be described as a classical mixture of coherent states which are averaged over all phases. This means that we can decompose the state into coherent states for the purpose of our calculation as long as we correctly average over phases at the end. We stress that these coherent states are just calculational tools: we never physically rely on them.

Referring to figure 5, we see that Alice combines her state ρ_a with a vacuum state, $|0\rangle$ at BS4. As discussed above, this input can be decomposed as $|0\rangle_9||\alpha|\,e^{i\phi_a}\rangle_{c10}$. If the transmittivity of BS4 is chosen to be $1/|\alpha|$, then it allows, on average, one particle to pass through and the output state is therefore $|e^{i\phi_a}\rangle_{c3}|i\sqrt{|\alpha|^2-1}\,e^{i\phi_a}\rangle_{c11}$ [20]. Similarly, Bob combines his reference state ρ_b with a vacuum state at a beam splitter (BS5) with transmittivity $1/|\alpha|$. This gives an output $|e^{i\phi_b}\rangle_{c8}|i\sqrt{|\alpha|^2-1}\,e^{i\phi_b}\rangle_{c14}$. Comparing with figure 1, we see that the inputs into BS2 and BS3 are the same as they were in scheme 1 and so the expression for S given by (13) also holds in this case.

The problem, however, is that, just like we found in scheme 1, we do not know the relative phase, $\Delta\phi$ of the reference states. This would suggest that we need to average over all values of $\Delta\phi$, which gives us $S < 2$, i.e. the CHSH inequality is not violated. However, the advantage of the present setup is that we do not need to perform this phase averaging because the crucial phase information is contained in the states on paths 11 and 14. This means that we could make a measurement on these states to determine what the value of $\Delta\phi$ is for each shot of the experiment. Although $\Delta\phi$ is not controllable, it can be determined on each run, which is all we need. In our scheme $\Delta\phi$ is found simply by combining paths 11 and 14 at a 50:50 beam splitter (BS6) and recording the number of particles, N_{15} and N_{16}, detected at ports 15 and 16, respectively. The expectation values for these are given by,

$$\langle N_{15}\rangle = (|\alpha|^2 - 1)\sin^2\left(\frac{\Delta\phi + \pi/2}{2}\right), \tag{25}$$

$$\langle N_{16}\rangle = (|\alpha|^2 - 1)\cos^2\left(\frac{\Delta\phi + \pi/2}{2}\right). \tag{26}$$

Combining these, we obtain,

$$\frac{\langle N_{16}\rangle - \langle N_{15}\rangle}{\langle N_{16}\rangle + \langle N_{15}\rangle} = 2\cos\left(\Delta\phi + \pi/2\right). \tag{27}$$

So long as the particle numbers detected are sufficiently large, this allows us to determine $\Delta\phi$ on each run.

In practice, on each experimental run Alice and Bob would record which of their detectors clicked and what setting they had chosen for their beam splitter. Then a measurement of $\Delta\phi$ would be performed at BS6. Of course, the measurements of $\Delta\phi$ could take place much later—after all of Alice and Bob's measurements have been completed. The results could then be 'grouped' into different ranges of $\Delta\phi$ and the correlations $E(\xi, \eta, \Delta\phi)$, $E(\xi + \pi/2, \eta, \Delta\phi)$, $E(\xi, \eta + \pi/2, \Delta\phi)$ and $E(\xi + \pi/2, \eta + \pi/2, \Delta\phi)$ could be determined for each $\Delta\phi$. This would enable us to determine the value of S as a function of $\Delta\phi$. A plot of how S varies with $\Delta\phi$ is shown in figure 6. We see in particular that in the shaded regions, $S > 2$. This violation of the CHSH inequality would provide an experimental signature of the nonlocality of a single particle.

One possible objection to this scheme is that the measurements are grouped together depending on the value of $\Delta\phi$. This could be viewed as a form of post-selection giving rise to difficulties similar to those discussed in section 3. However, the present scheme is different for two reasons. Firstly, no measurement outcome is discarded. Secondly, $\Delta\phi$ is not a parameter of the nonlocal state under investigation, but rather a setting of the measurement process. In principle, Alice and Bob can control the value of $\Delta\phi$, whereas they could not control unfavorable measurement outcomes in section 3. An analogous case is that of Alice and Bob
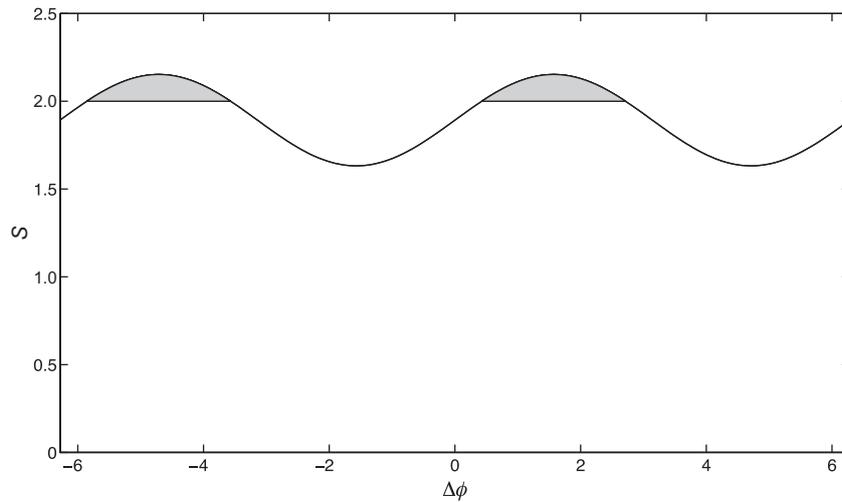
Figure 6. This graph shows how S varies with $\Delta\phi$ as given by equation (19) for $\xi - \eta = 3\pi/4$. A violation of the CHSH inequality is evident in the shaded regions where $S > 2$.

performing a standard Bell experiment with pairs of entangled spin-half particles where their detector orientations are randomly set for each measurement. Only after all the measurements are made are Alice and Bob given a 'printout' of what the orientations were in each run. In this case, they can still extract data after the event that demonstrates violations of Bell's inequality. Similarly in the present scheme, Alice and Bob's ignorance of their detector settings $\Delta\phi$ until after the measurement has been made need not undermine their ability to rule out local models.

Finally, we note that this scheme is not restricted to Alice and Bob using reference states of the form of (24). In particular, if Alice and Bob were to each use number states, $|N_a\rangle$ and $|N_b\rangle$, the scheme would work in the same way so long as they picked the transmittivities of BS2 and BS3 to be $1/\sqrt{N_a}$ and $1/\sqrt{N_b}$, respectively. By extension, we see that general mixtures of number states of the form,

$$\rho = \sum_{n=0}^{\infty} P(n) |n\rangle\langle n| \tag{28}$$

should also work. The only potential problem with this is that Alice and Bob need to set the transmittivities of their beam splitters according to the number of incident particles. This is to ensure that they skim off a state with a mean amplitude of one. In practice, so long as the width of the distribution, $P(n)$, is small with respect to the mean number of particles, this should be able to be achieved to a very good approximation.

6. Conclusions

We have presented three schemes for observing violations of the CHSH inequality due to a single particle. In general, we have seen that these schemes all require additional reference particles to be added so that Alice and Bob can each make meaningful measurements. The key is that these additional particles are introduced in such a way that they do not contribute any nonlocality to the system.

In the first scheme, Alice and Bob each used an independently created coherent state as their reference. We found that, since there was no fixed relationship between the phases of the reference states, the evidence for nonlocality was washed out and it was not possible to violate the CHSH inequality. The second scheme resolved this issue by using the two outputs from a single beam splitter as the reference states. This ensured that there was a fixed phase relationship between them. As expected, this allowed for a violation of the CHSH inequality. Indeed this idea is similar to experimental schemes currently used to test single particle nonlocality [23]. One possible problem with the second scheme, however, is that it relies on a global operation to create the two reference states. This means that, in general, we cannot guarantee that the reference states are completely separable and will not be responsible for the observed violation of the CHSH inequality.

In the third scheme, we overcame this concern by ensuring that Alice and Bob each had independently created mixed states. They then skimmed off part of these states to be used as their reference state and the remaining part was used later to determine the relative phase between their references. Although the relative phase could not be controlled nor predicted in advance, it was knowable on every experimental run. This allows S to be determined as a function of $\Delta\phi$. A theoretical plot is shown in figure 6, which predicts that there are values of $\Delta\phi$ where $S > 2$, i.e. the CHSH inequality is violated. Such a scheme has intriguing prospects for advancing our understanding of the issue of single particle entanglement. Importantly, this is equally applicable to atoms as well as photons as all the employed techniques, such as beam splitters with variable reflectivities, are achievable for both.

Acknowledgments

References

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