Relative number squeezing in Bose-Einstein condensates

J. A. Dunningham and K. Burnett
Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom

Mark Edwards
Department of Physics, Georgia Southern University, Statesboro, Georgia 30460-8031

(Received 4 January 2001; published 30 May 2001)

We describe a procedure for creating pairs of condensates with appreciable relative number squeezing. We show that our procedure is relatively robust against the effects of loss and may therefore prove to be a practical way of generating such states. We use a quantum simulation for small numbers that enables us to validate a semiclassical model. This is used to predict the scaling for large numbers.

DOI: 10.1103/PhysRevA.64.015601 PACS number(s): 03.75.Fi, 42.50.Dv, 42.50.Ct

I. INTRODUCTION

Number correlated pairs of Bose-Einstein condensates are a powerful quantum resource. They can, for example, be used to generate states with Heisenberg-limited relative phase [1,2] of great importance in interferometry [3]. In this paper we demonstrate how pairs of condensates may be created with relative number squeezing. The method we outline does not depend strongly on knowing the number of atoms in the initial condensate and works even in the presence of large losses. It may prove to be a practical technique for generating this resource. It relies on choosing the right balance of coupling between condensates and interactions between atoms in each condensate.

II. QUANTUM ANALYSIS

The model we use is similar to one proposed by Cirac *et al*. for creating superposition states of Bose condensates [4]. We consider two condensate modes represented by the annihilation operators a and b. Mode a is initially in a number state $|N\rangle$ and b is initially in the vacuum state $|0\rangle$. We then couple these two modes with resonant Raman pulses to create a pair of condensates with a relative phase defined to the standard quantum limit. If this step is very fast compared with the time scale of the nonlinear evolution, we can ignore the effects of interactions. After a quarter Raman cycle, the state is given by

$$|\psi\rangle = e^{i\pi(a^\dagger b + b^\dagger a)/4}|N\rangle_a|0\rangle_b = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\sqrt{\frac{N!}{k!\,(N-k)!}}e^{-i\pi k/2}|k\rangle_a|N-k\rangle_b. \quad (1)$$

This is a superposition of states with different relative numbers of atoms in the two modes. The relative phase $\Delta\theta$ of this state scales with number in the same way as the broken symmetry state that can be produced by measurement [2], $\Delta\theta \sim 1/\sqrt{N}$. The number correlation between the modes in Eq. (1) is weak and we would like to squeeze the relative number distribution.

To do this, we first apply a far off resonant light pulse to mode a to shift the phase by $-\pi/2$ [5]. The state after this step is

$$|\psi\rangle = \frac{1}{\sqrt{2^N}}\sum_{k=0}^{N}\sqrt{\frac{N!}{k!\,(N-k)!}}e^{-i\pi k}|k\rangle_a|N-k\rangle_b, \quad (2)$$

which we take to be the initial state of our squeezing procedure. In this state, the two modes a and b have zero mean relative phase.

Next we couple the two condensate modes with resonant Raman pulses, which is equivalent to Josephson coupling the modes, and we allow them to evolve under the influence of the nonlinear interactions. The Hamiltonian for this evolution is taken to be

$$H = U(a^{\dagger 2}a^2 + b^{\dagger 2}b^2) + \Gamma(a^\dagger b + b^\dagger a), \quad (3)$$

where U is the interatomic interaction strength which we take to be the same for each mode for simplicity, and Γ is the coupling strength. We take the trap frequencies to be the same for the two modes and have removed them by transforming to a rotating frame. The parameter U depends on the intrinsic atomic interactions and the shape of the traps. This can be tuned in principle [6], but for a given experiment is fixed. The coupling rate Γ can be controlled by the experimentalist by varying the strength of the coupling laser.

In Fig. 1, we have plotted how the relative number distribution of Eq. (2) evolves for the parameters $N=100$, $U=0.5$, and $\Gamma=1$ where, for convenience, U and Γ are unitless quantities scaled by Γ. In Fig. 1(a), we plot the variance of $N_a - N_b$ as a function of Ωt, where $\Omega = 2\sqrt{\Gamma UN + \Gamma^2}$. This is a measure of the width of the relative number distribution and we see that it undergoes oscillations. To begin with, the distribution gets narrower with time and, for the parameters used here, the maximum squeezing occurs at $\Omega t = 1.67$. This is precisely the result that we want and it is encouraging how strong the squeezing is. For the present parameters, a reduction in the variance by a factor of 25 is observed.

In Fig. 1(b), we plot the number distribution of mode a at the optimum squeezing time. This is identical to the number distribution of mode b. The original number distribution is

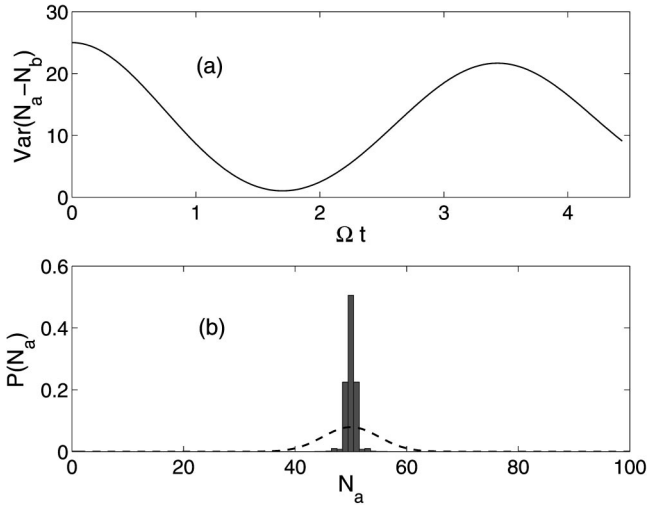

FIG. 1. (a) Time variation of the variance of $(N_a - N_b)$ as a function of Ωt, where $\Omega = 2\sqrt{\Gamma UN + \Gamma^2}$. (b) Number distribution of mode a at the optimum squeezing time.

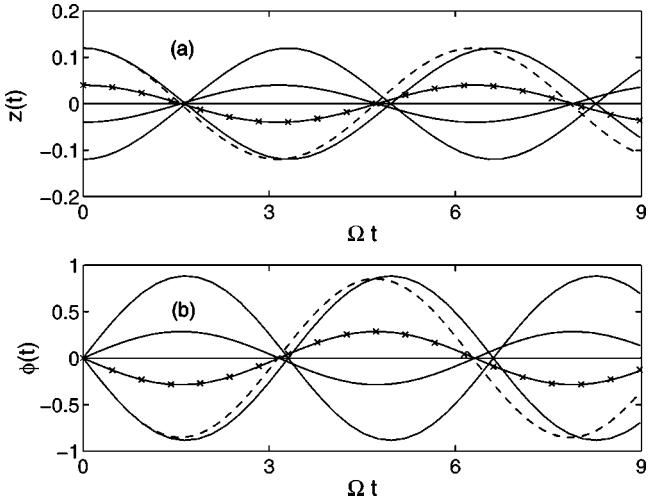

FIG. 2. Solutions of the semiclassical equations for different initial conditions as a function of Ωt. The $z(t)$ solutions are shown as the full curves in (a) and the corresponding $\phi(t)$ solutions are shown as the full curves in (b). The dashed and crossed curves are corresponding $z(t)$ and $\phi(t)$ solutions of the linearized equations for two different initial conditions.

also shown as a dashed line for comparison and we see that the modes are strongly number squeezed around the mean value of $N/2$. In other words, the two modes are strongly number correlated.

III. SEMICLASSICAL MODEL

In order to understand these results, we perform a semiclassical analysis of the system [7,8]. Our results do not depend on this, but we believe that it gives considerable further insight into them. The Hamiltonian for the system is given by Eq. (3), which allows us to write down the equations of motion for the operators,

$$i\dot{a}(t) = 2Ua^\dagger aa + \Gamma b, \tag{4}$$

$$i\dot{b}(t) = 2Ub^\dagger bb + \Gamma a. \tag{5}$$

We can make a semiclassical approximation by making the replacements

$$a(t) = \sqrt{N_a(t)}\, e^{i\theta_a(t)}, \tag{6}$$

$$b(t) = \sqrt{N_b(t)}\, e^{i\theta_b(t)}, \tag{7}$$

where $N_{a,b}$ corresponds to the number of atoms in modes a,b and $\theta_{a,b}$ corresponds to the phase of a,b. This is a reasonable approach to take since, if we look at the form of Eq. (2), we see that each mode has the form of a coherent state and so we can replace the operators with complex numbers containing the mean amplitude and mean phase of each mode.

Next we define the new quantities of fractional population imbalance,

$$z(t) \equiv \frac{\langle b^\dagger b\rangle - \langle a^\dagger a\rangle}{\langle b^\dagger b\rangle + \langle a^\dagger a\rangle} = \frac{N_b(t) - N_a(t)}{N}, \tag{8}$$

where $N = N_a + N_b$ is the constant total number of atoms, and the relative phase

$$\begin{aligned}\phi(t) &\equiv \arg\{\langle a^\dagger b\rangle\} - \pi \\ &= \theta_b(t) - \theta_a(t) - \pi.\end{aligned} \tag{9}$$

We introduce this phase shift since we shall see that it is convenient if the initial value of Eq. (9) is zero when we come to linearize the system. This transformation in no way changes our results. We can derive semiclassical equations of motion for this system in terms of these new quantities [7],

$$\dot{z}(t) = 2\Gamma\sqrt{1 - z^2(t)}\,\sin[\phi(t)], \tag{10}$$

$$\dot{\phi}(t) = -2UNz(t) - \frac{2\Gamma z(t)}{\sqrt{1 - z^2(t)}}\cos[\phi(t)]. \tag{11}$$

In Fig. 2, the solid lines show numerical solutions of Eqs. (10) and (11) for different initial conditions. We set the parameters to be the same as for the quantum calculation, $U = 0.5$, $\Gamma = 1$, and $N = 100$. Each trajectory corresponds to $z(0) \in \{-0.12, -0.04, 0, 0.04, 0.12\}$ and, since the initial state has zero relative phase between the modes [see Eq. (2)], we take $\phi(0) = 0$ for each trajectory.

The quantum state can be thought of loosely as a superposition of these classical realizations. Here we interpret each trajectory to represent part of the state and, in particular, a few adjacent terms in Eq. (2), i.e., terms with similar numbers of atoms in mode a. The way that the state is split up into trajectories is arbitrary and does not affect our results. The parameters z and ϕ now respectively represent the mean number difference between modes and the mean phase for

each of these parts. The evolution of the whole state can be seen by observing the evolution of each of its parts or trajectories.

We notice that z and ϕ both undergo oscillatory motion, which is what we observed in the full calculation. It is important, to note that at certain times z vanishes, independently of the initial value of z. We see in Fig. 2(a) that the first focus is sharp and that subsequent focus points become more and more blurred. This means that if we started with an initial state that was a superposition of different values of z, and allowed this to evolve under the Hamiltonian (3), after some time the state would be strongly squeezed about $z=0$. The predictions of the semiclassical model agree well with the full calculation.

First, we would like to understand the times at which the squeezing is optimized. To do this, we assume that $|z(t)| \ll 1$ and $\sin[\phi(t)] \approx \phi(t)$. The second assumption is accurate to within a few percent for $\phi(t) < 0.5$. We will justify these assumptions later.

Linearizing Eqs. (10) and (11) allows us to write

$$\dot{z}(t) \approx 2\Gamma \phi(t), \tag{12}$$

$$\dot{\phi}(t) \approx -2(UN+\Gamma) z(t). \tag{13}$$

These equations are very straightforward to solve and give the result

$$z(t) \approx z(0)\cos(\Omega t), \tag{14}$$

$$\phi(t) \approx -\frac{\Omega}{2\Gamma} z(0)\sin(\Omega t), \tag{15}$$

where $\Omega = 2\sqrt{\Gamma UN + \Gamma^2}$. We see that the conditions $|z(t)| \ll 1$ and $|\phi(t)| < 0.5$ hold for $|z(0)| \ll 1$ and $(\Omega/\Gamma) z(0) < 1$.

If we assume that our initial relative number distribution is Gaussian (as is the case for the full calculation) then $|z(0)| < \sim 1/2\sqrt{N}$, and $(\Omega/\Gamma) z(0) < \sim \sqrt{U/\Gamma}$. This means that the system is well described by the linear equations (12) and (13) for $N \gg 1$ and $U/\Gamma < 1$. These conditions are satisfied by the parameters we use here.

In Fig. 2, we have plotted the solutions of (a) $z(t)$ and (b) $\phi(t)$ for $(z(0), \phi(0)) = (0.12, 0)$ (dashed curve) and $(z(0), \phi(0)) = (0.04, 0)$ (crossed curve). For the crossed solution, $\max[z(t)] = 0.04$ and $\max[\phi(t)] = 0.28$ and our two conditions are satisfied. We see that this approximate solution is very close to the solution of the full equations and it is hard to distinguish the two trajectories in Fig. 2. For the dashed line, $\max[z(t)] = 0.12$ and $\max[\phi(t)] = 0.86$, the two conditions not satisfied, and we see that this approximate solution is not very good for long times. It does, however, still predict the first squeezing time quite well. We can conclude that the linearized equations should provide a good description of the full model and we can write down an analytical approximation for the optimal squeezing time,

$$t_{sq} = \frac{\pi}{2\Omega} = \frac{\pi}{4\sqrt{\Gamma(UN+\Gamma)}}. \tag{16}$$

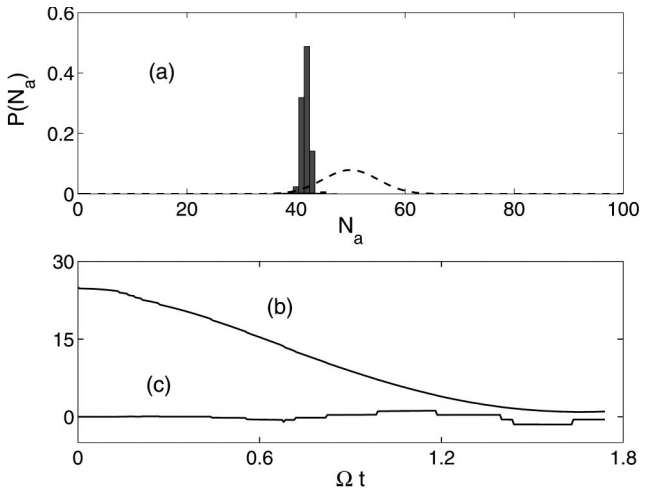

FIG. 3. (a) Number distribution of mode a at the optimum squeezing time (bar graph) compared with the number distribution of the initial state (dashed curve). (b) Variance of $(N_a - N_b)$ as a function of Ωt. (c) Mean value of $(N_a - N_b)$ as a function of Ωt. For this simulation, nine atoms were lost from a and eight from b.

This is in agreement with the result shown in Fig. 2 and agrees to within a few percent with the quantum prediction of $\Omega t = 1.67$.

IV. EFFECT OF LOSS

In any realistic physical system, there will be some degree of loss due, for example, to collisions. In this section we consider how loss affects our ability to create states with squeezed relative number by including random loss from modes a and b in the quantum model. The results for our simulation are shown in Fig. 3.

We consider a system with the same parameters as before, $N = 100$ and $U/\Gamma = 0.5$, and consider the rate of damping to be the same for each mode. In Fig. 3(a) we show the number distribution for mode a at the optimum squeezing time and compare it to the initial distribution. For this particular trajectory, nine atoms were lost from a and eight from b. We see that the mean number of atoms in a is reduced to around 41, as we might expect, but interestingly the distribution is still strongly squeezed.

In fact, the variance of this number distribution is as small as for the lossless case. The variance of the two modes must be identical since the total number of atoms is equal to some particular value. This means that even for large losses (in this case 17%) the relative number distribution $\mathrm{Var}(N_a - N_b)$ will be strongly squeezed by this method. In realistic cases we would expect the loss to be much less than this.

The variance of the number distribution of mode a as a function of Ωt is shown in Fig. 3(b). It exhibits the same behavior as the lossless case and an almost identical optimum squeezing time. In Fig. 3(c) we show the time variation of the difference of the mean number of atoms in modes a and b. Owing to the loss, this is not necessarily zero and this effect introduces some additional uncertainty into the number correlation between the two modes.

An estimate of the width of the distribution of the differ-

ence in mean number is given by $\sqrt{N_{\rm loss}}$, where $N_{\rm loss}$ is the total number of atoms lost. We can write

$$N_{\rm loss} \approx \gamma N t = \gamma \sqrt{\frac{N}{\Gamma U}}, \tag{17}$$

where γ is the rate of loss per atom per unit time and we have put in the optimum squeezing time. This means that our uncertainty in the difference of the mean numbers scales as $\sqrt{\gamma} N^{1/4}$. So, for large N or small γ, this is much smaller than the uncertainty in the number difference of the original Gaussian state, which scales as $\sqrt{N}$. This method for creating number correlated condensates therefore works even in the presence of appreciable losses during the preparation. We have shown that dissipation has little effect on the variance of the squeezed number state and that the relative mean number uncertainty that is introduced is much smaller than the number uncertainty of the original state.

We can understand this result with reference to Fig. 2(a). If we imagine a solution following one of the trajectories in Fig. 2(a), the loss of a few atoms will simply move us to a different trajectory. Since all trajectories focus at the same time, our initial reaction is that it does not matter if atoms are lost, and there is still optimum relative number squeezing at $t = \pi/2\Omega$. The reason that the number correlation is not as good as in the lossless case is that when atoms are lost, although the z solution is moved to another trajectory, the ϕ solution is unchanged. From Eq. (13) we see that the phase of this new trajectory differs from what it should be for perfect focusing by an amount proportional to the amount z is changed by the loss. From Eq. (8), we see that the change in z scales as $1/N$ so, for large N, the disruption to the phase of the trajectory is small even for significant losses. This is why this method is relatively robust to loss.

So far things look good for the experimental feasibility of this scheme. However, one practical difficulty that arises is that the optimum squeezing time depends on the number of atoms in the system. This suggests that to implement optimum squeezing we would need to know how many atoms were in the system to begin with. It is very unlikely that we would have this information. Fortunately, the squeezing is relatively insensitive to evolution time. We can see this in Fig. 1. For example, even if we underestimated N by a factor of 2 and so allowed the system to evolve for $t = t_{\rm opt}/\sqrt{2}$, where $t_{\rm opt}$ is the optimum squeezing time, the final relative number distribution would still be strongly squeezed. For the parameters used here, the final variance would be about 4.5, which is about 5.5 times smaller than in the original state.

For improved correlation, we can imagine a two (or more) step process. In this we could estimate the number of atoms, allow the system to evolve for the estimated optimum squeezing time, and then perform a destructive measurement of the number of atoms in one condensate. This would give us an accurate estimate of the number of atoms in the other condensate without destroying it. We could then use this condensate as the starting point for a second relative number squeezing process. This time, however, our improved knowledge of the total number of atoms in the system would allow us to predict the optimum squeezing time more accurately and so create better number correlated condensates.

ACKNOWLEDGMENTS

We would like to thank M. Kasevich and W. D. Phillips for helpful discussions. This work was financially supported by the British Council, the United Kingdom EPSRC, and the EU, under the TMR network ''Coherent Matter Wave Interactions,'' Contract No. ERB-FMRX-CT-0002.

[1] P. Bouyer and M.A. Kasevich, Phys. Rev. A **56**, R1083 (1997).
[2] J.A. Dunningham and K. Burnett, Phys. Rev. A **61**, 065601 (2000).
[3] M.J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[4] J.I. Cirac, M. Lewenstein, K. Mølmer, and P. Zoller, Phys. Rev. A **57**, 1208 (1998).
[5] D. Gordon and C.M. Savage, Phys. Rev. A **59**, 4623 (1999).
[6] S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **85**, 1795 (2000).
[7] S. Raghavan, A. Smerzi, S. Fantoni, and S.R. Shenoy, Phys. Rev. A **59**, 620 (1999).
[8] G.J. Milburn, J. Corney, E.M. Wright, and D.F. Walls, Phys. Rev. A **55**, 4318 (1997).